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Analysis Of A Coupled Quadratic Volterra-Stieltjes Integral Equation System

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Abstract:
This paper aims to establish existence results for a coupled system of nonlinear quadratic integral equations of the Volterra-Stieltjes type in the space of continuous functions over a closed bounded interval. The existence of solutions is demonstrated using the Schauder fixed point principle. This approach enables us to derive existence theorems under broad and general conditions.

Key words: Coupled system, nonlinear quadratic integral equation, function of bounded variation, fractional order

1. Introduction

Integral equations play a crucical role in modeling various phenomena and events in applied sciences. This field has witnessed significant development through the use of functional analysis, topology, and fixed point theory (see[1, 9, 10, 11, 12, 15]). Among these advancements, Volterra-Stieltjes integral equations have attracted considerable attention, with several studies dedicated to their analysis (see [5, 7, 16, 17, 18, 19]).

The primary objective of this paper is to examine the solvability of a coupled system of quadratic Volterra-Stieltjes integral equations.

Give the importance of the Stieltjes integral in this context; we rely on the definitions and properties introduced by Banas` (see [2, 3]).

Additionally, interest in studying such coupled system has been sparked by previous research on related topics.

In this study, we establish some existence theorems for a coupled system of quadratic Volterra-Stieltjes integral equations, which encompass various typs of Volterra integral equations as special cases. Our proof is based on the fixed-point principle, enabling us to derive existence results under broad and flexible assumptions.

Throughout this paper, let $\boldsymbol{I} = [\boldsymbol{0}, \boldsymbol{T}]$ and $\boldsymbol{X}$ be the Banach space of all ordered pairs $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} = \boldsymbol{C}(\boldsymbol{I}) \times \boldsymbol{C}(\boldsymbol{I})$, with the norm

$$\|(x,y)\|_X = max\,\{\|x\|_C, \|y\|_C\},$$

where

$$\|x\|_C = \sup_{t\in I}|x|, \|y\|_C = \sup_{t\in I}|y|,$$

It is clear that $(X, \|(x,y)\|_X)$ is Banach space.

Now, we shall present some auxiliary properties of fractional calculus that will be need in this work.

Definition 1. The Riemann-Liouville of a fractional integral of the function $f \in L\_1\ (I)$ of order $\alpha \in R^+$ is defined by

$$I_a^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

and when $h = 0$, we have $I^\alpha f(t) = I_a^\alpha f(t)$.

Definition 2. The (Caputo) fractional order derivative $D^\alpha, \alpha \in (0,1]$ of the absolutely continuous function g is defined as

$$D_a^\alpha g(t) = I_a^{1-\alpha} \frac{d}{dt} g(t), \qquad t \in [a,b].$$

For further properties of fractional calculus operator (See [20], [21], [22] and [23]).

2. Preliminaries

In this section, we study the solvability of the coupled system of nonlinear quadratic integral equations of Volterra-Stieltjes type having the form

$$x(t) = h_1(t) + f_1(t, y(\omega_1(t)) \int_0^t u_1(t,s,y(\omega_1(s))d_s g_1(t,s), \qquad t \in I$$

(1)

$$y(t) = h_2(t) + f_{12}(t, y(\omega_2(t)) \int_0^t u_2(t,s,y(\omega_2(s))d_s g_2(t,s), \qquad t \in I$$

Our goal is to show that system (1) has at least one solution in the space $\boldsymbol{X}$. For our further purposes we denote by $\triangle$ the triangle $\triangle = \{(\boldsymbol{t}, \boldsymbol{s}): \boldsymbol{0} \leq \boldsymbol{s} \leq \boldsymbol{t} \leq \boldsymbol{T}\}$.

In our investigations, we give some assumptions which are needed throughout this paper.

(i) $\boldsymbol{h_i}: \boldsymbol{I} \to \boldsymbol{R}, (\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2})$ are continuous on $\boldsymbol{I}$. There are constants $\boldsymbol{h_i}$, where $\boldsymbol{h_i} = \sup_{t \in I} |\boldsymbol{h_i}(\boldsymbol{t})|$.

(ii) $\boldsymbol{fi} : \boldsymbol{I} \times \boldsymbol{R} \to \boldsymbol{R}, (\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2})$ are continuous functions and there exist continuous functions
$\boldsymbol{m_i}(\boldsymbol{t}) : \boldsymbol{I} \to \boldsymbol{I}$ such that

$$|\boldsymbol{f_i}(\boldsymbol{t}, \boldsymbol{x}) - \boldsymbol{f_i}(\boldsymbol{t}, \boldsymbol{y})| \leq \boldsymbol{m_i}(\boldsymbol{t})|\boldsymbol{x} - \boldsymbol{y}|,$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}$ and $\boldsymbol{t} \in \boldsymbol{I}$. Moreover, we put $\boldsymbol{m_i} = \boldsymbol{max}\{\boldsymbol{m_i}(\boldsymbol{t})\ \boldsymbol{t} \in \boldsymbol{I},\}$.

(iii) $\boldsymbol{u_i}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}) : \triangle \times \boldsymbol{R} \to \boldsymbol{R}, (\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2})$ are continuous functions and there exist continuous functions $\boldsymbol{n_i}(\boldsymbol{t}, \boldsymbol{s}) : \triangle \to \boldsymbol{I}$, and continuous and nondecreasing functions $\boldsymbol{\varphi_i} : \boldsymbol{R}+ \to \boldsymbol{R}+$, such that

$$|\boldsymbol{u_i}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x})| \leq \boldsymbol{n_i}(\boldsymbol{t}, \boldsymbol{s})\boldsymbol{\varphi_i}(|\boldsymbol{x}|),$$

for all $(\boldsymbol{t}, \boldsymbol{s}) \in \triangle$ and $\boldsymbol{x} \in \boldsymbol{R}$. Moreover, we put $\boldsymbol{n_i} = \boldsymbol{max}\{\boldsymbol{n_i}(\boldsymbol{t}, \boldsymbol{s})\ \boldsymbol{t}, \boldsymbol{s} \in \boldsymbol{I}\}$.

(iv) $\omega_i : I \to I$ are continuous, $(\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2})$.

(v) Functions $\boldsymbol{s} \to \boldsymbol{g_i}(\boldsymbol{t}, \boldsymbol{s})$ are of bounded variation on $[\boldsymbol{0}, \boldsymbol{t}]$ for each $\boldsymbol{t} \in \boldsymbol{I}, \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}$.

(vi) Functions $\boldsymbol{g_i}$ are continuous on the triangle $\triangle$ and $\boldsymbol{g_i}(\boldsymbol{t}, \boldsymbol{0}) = \boldsymbol{0}$ for $\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}$.

(vii) $\boldsymbol{g_i}(\boldsymbol{t}, \boldsymbol{s}) = \boldsymbol{g_i} : \Delta_i \to \boldsymbol{R}, \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}$ and for all $\boldsymbol{t_1}, \boldsymbol{t_2} \in \boldsymbol{I}$ with $\boldsymbol{t_1} < \boldsymbol{t_2}$, the functions $\boldsymbol{s} \to \boldsymbol{g_i}(\boldsymbol{t_2}, \boldsymbol{s}) - \boldsymbol{g_i}(\boldsymbol{t_1}, \boldsymbol{s})$ are nondecreasing on I.

(viii) For any $\boldsymbol{\epsilon} > \boldsymbol{0}$ there exists $\boldsymbol{\delta} > \boldsymbol{0}$ such that, for all$\boldsymbol{t_1}; \boldsymbol{t_2} \in \boldsymbol{I}$ such that $\boldsymbol{t_1} < \boldsymbol{t_2}$ and $\boldsymbol{t_2} - \boldsymbol{t_1} \leq \boldsymbol{\delta}$

the following inequality holds

$$\bigvee_{s=0}^{t}[g_i(t_2,s)-g_i(t_1,s)] \le \epsilon, \quad i=1,2.$$

Obviously, we will assume that $g_i, (i = 1,2)$ satisfy assumptions $(iv)-(vii)$. For our purposes, we will need the following lemmas.

Lemma 1. [6] Under assumptions (v)-(viii), The functions $z \to \bigvee_{s=0}^{z} g_i(t,s)$ are continuous on $[0,t]$ for any $t \in I$ $(i=1,2)$.

Lemma 2. [6] Let assumptions (v)-(viii) be satisfied. Then, for arbitrary fixed number $0 < t_2 \in I$
and for any $\epsilon > 0$, there exists $\delta > 0$ such that if $t_1 \in I$; $t_1 < t_2$ and $t_2 - t_1 \le \delta$ then $\bigvee_{s=t_1}^{t_2} g_i(t_2,s) \le \epsilon$. $(i=1,2)$.

Further, let us observe that based on Lemma 1 we infer that there exists finite positive constants
K_i, such that

$$K_i = sup\left\{\bigvee_{s=0}^{t} g_i(t,s) : t \in [0,T]\right\}.$$

where $T > 0$ is arbitrarily fixed $(i = 1,2)$.
We now introduce some functions that will be useful in our further studies:

$$W_i = sup\left\{\bigvee_{s=0}^{t_2}\big(g_i(t_2,s)-g_i(t_1,s)\big) : t_1, t_1 \in I, t_1 < t_2 \le \epsilon, i = 1,2 \right\}.$$

In what follows let us denote by F_i the constant defined by the formula:

$$F_i = sup\{|f_i(t,0)| : t \in I, i = 1,2\}.$$

Now, we are in position to present tha main result of the paper.

3. Main Result

Defined the operator by

$$T(x,y)(t) = \left(T_1 y(t), T_2 x(t)\right),$$

where

$$T_1 x(t) = \; h_1(t) + f_1(t, y(\omega_1(t)) \int_0^t u_1(s, y(\omega_1(s)) d_s g_1(t,s)\,, \;\; t \in I$$
(2)

$$T_2 y(t) = \; h_2(t) + f_{12}(t, y(\omega_2(t)) \int_0^t u_2(s, y(\omega_2(s)) d_s g_2(t,s)\,, \qquad t \in I$$

Theorem 1. Let assumptions (i)-(vii) be satisfied. Then the coupled system of quadratic Volterra-Stieltjes integral equations (1) has at least one solution (x,y) belonging to the space X.

Proof. We prove a few results concerning the continuity and compactness of these operators in the space of continuous functions.
Define

$$V = \left\{v = \left(x(t), y(t)\right) : \left(x(t), y(t)\right) \in X : \; \|(x,y)\|_X \le r\right\}.$$

For $(x,y) \in V$, and define the operator T map V into V, we have

$$|T_1 y(t)| \le |h_1| + |f_1(t, y(\omega_1(t))| \int_0^t |u_1(t,s,y(\omega_1(s))||d_s g_1(t,s)|$$

$$\le \|h_1\| + [m_i|y(t)| + |f_1(t,0)|] \int_0^t n_1(t,s)\varphi_1(|y(s)|) d_s (\vee_{P=0}^t g_1(t,P),$$

$$\le \|h_1\| + [\|y\| m_1 + F_1] n_1 \varphi_1(\|y\|)(\vee_{p=0}^t g_1(t,p),$$

$$\|T_1 y\| \le \|h_1\| + [r_1 m_1 + F_1] n_1 \varphi_1(r_1) \sup_{t\in I}(\vee_{p=0}^t g_1(t,p)).$$

Hence, we get

$$\|T_1 y\| \le \|h_1\| + K_1[m_1 r_1 + F_1] n_1 \varphi_1(r_1).$$

From the last estimate we deduce that $r_1 = \frac{\|h_1\| + F_1 K_1 n_1 \varphi_1(r_1)}{1 - m_1 n_1 K_1 \varphi_1(r_1)}$.

By a similar way, we obtain

$$\|T_2y\| \leq \|h_2\| + K_2[m_2r_2 + F_2]n_2\varphi_2(r_2).r_2 = \frac{\|h_2\| + F_2K_2n_2\varphi_2(r_2)}{1 - m_2n_2K_2\varphi_2(r_2)}.$$

Therefore

$$\|Tv\|_X = \|T(x,y)\|_X = \|T_1y, T_2x\|_X$$
$$\leq \max_{t\in I}\{\|T_1y\|_C, \|T_2y\|_C\} = r.$$

Thus for every $v = (x,y) \in V$, we have $Tv \in V$ and hence $TV \subset V$, ($i.e\ T : V \to V$). This means that the functions of TU are uniformly bounded on I, it is clear that the set V is nonempty, bounded, closed and convex. Now, we need to show that the set TV is relatively compact.
For $v = (x,y) \in V$, for all $\epsilon > 0, \delta > 0$, and for each $t_1, t_2 \in I$(without loss of generality assume that $t_1 < t_2$, such that $|t_2 - t_1| < \delta$, we have

$$|T_1y(t_2) - T_1y(t_1)|$$

$$= \Big|h_1(t_2) - h_1(t_1) + f_1(t_2, y(\omega_1(t_2))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)$$

$$- f_1(t_1, y(\omega_1(t_1))) \int_0^{t_1} u_1\left(t_1, s, y(\omega_1(s))\right) d_sg_1(t_1, s)\Big|$$

$$\leq |h_1(t_2) - h_1(t_1)| + |f_1(t_2, y(\omega_1(t_2))) - f_1(t_2, y(\omega_1(t_1)))|$$

$$.\ \left|\int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\right|$$

$$+ \left|f_1(t_2, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\right.$$

$$- \left.|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\right|$$

$$+ \left|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\right.$$

$$- \left.|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_1, s)\right|$$

$$+\left|f_1(t_1,y(\omega_1(t_1)))\int_0^{t_2}u_1\left(t_2,s,y(\omega_1(s))\right)d_sg_1(t_1,s)\right.$$

$$-\left.f_1(t_1,y(\omega_1(t_1)))\int_0^{t_2}u_1\left(t_1,s,y(\omega_1(s))\right)d_sg_1(t_1,s)\right|$$

$$+\left|f_1(t_1,y(\omega_1(t_1)))\int_0^{t_2}u_1\left(t_1,s,y(\omega_1(s))\right)d_sg_1(t_1,s)\right.$$

$$-\left.f_1(t_1,y(\omega_1(t_1)))\int_0^{t_1}u_1\left(t_1,s,y(\omega_1(s))\right)d_sg_1(t_1,s)\right|$$

$$\leq \aleph(h_1,\epsilon)+m_1(t_2)|y(t_2)-y(t_1)|\int_0^{t_2}\left|u_1(t_2,s,y(\omega_1(s)))\right|d_s\left(\bigvee_{p=0}^{s}g_1(t_2,p)\right)$$

$$+\left|f_1\left(t_2,y(t_1)\right)-f_1\left(t_1,y(t_1)\right)\right|\int_0^{t_2}\left|u_1\left(t_2,s,y(\omega_1(s))\right)\right|d_s\left(\bigvee_{p=0}^{s}g_1(t_2,p)\right)$$

$$+\left|f_1\left(t_1,y(t_1)\right)\right|\int_0^{t_2}\left|u_1\left(t_2,s,y(\omega_1(s))\right)\right|d_s\left(\bigvee_{p=0}^{s}[g_1(t_2,p)-g_1(t_1,p)]\right)$$

$$+\left|f_1\left(t_1,y(t_1)\right)\right|\int_0^{t_2}\left|u_1\left(t_2,s,y(\omega_1(s))\right)-u_1\left(t_1,s,y(\omega_1(s))\right)\right|d_s\left(\bigvee_{p=0}^{s}g_1(t_1,p)\right)$$

$$+\left|f_1\left(t_1,y(t_1)\right)\right|\int_{t_1}^{t_2}\left|u_1(t_1,s,y(\omega_1(s)))\right|d_s\left(\bigvee_{p=0}^{s}g_1(t_1,p)\right)$$

$$\leq \aleph(h_1,\epsilon)+m_1(t_2)\aleph(y,\epsilon)\int_0^{t_2}n_1(t_2,s)\varphi_1(|y(s)|)d_s\left(\bigvee_{p=0}^{s}g_1(t_2,p)\right)$$

$$+\aleph_{f1}(\epsilon)\int_0^{t_2}n_1(t_2,s)\varphi_1(|y(s)|)d_s\left(\bigvee_{p=0}^{s}g_1(t_2,p)\right)$$

$$+[m_1(t_1)|y(t_1)|+|f_1(t_1,0)|]\int_0^{t_2}n_1(t_2,s)\varphi_1(|y(s)|)d_s\left(\bigvee_{p=0}^{s}[g_1(t_2,p)-g_1(t_1,p)]\right)$$

$$+[m_1(t_1)|y(t_1)|+|f_1(t_1,0)|]\int_0^{t_2}\aleph_{u1}(\epsilon)d_s\left(\bigvee_{p=0}^{s}g_1(t_1,p)\right)$$

$+[m_1\ (t_1)|y(t_1)| + |f_1(t_1,0)|] \int_{t_1}^{t_2} n_1(t_1,s)\varphi_1(|\ y(s)|)d_s\left(\vee_{p=0}^{s} g_1(t_1,p)\right)$

Where

$$\aleph(h_i,\epsilon) = sup\,\{|h_1(t_2) - h_1(t_1)|:\ t_1,t_2 \in\ I, |t_2\ -\ t_1| <\ \epsilon, i = 1,2\ \},$$

$$\aleph_{fi}(\epsilon) = sup\,\{|f_i(t_2,v) - f_i(t_1,v)|:\ \ t_1,t_2 \in\ I, |t_2\ -\ t_1| < \epsilon, v \in R, i = 1,2\},$$

$$\aleph_{ui}(\epsilon) = \left\{\left|u_1\big(t_2,s,v(s)\big) - u_2(t_1,s,v(s))\right|:\ t_1,t_2 \in\ I, |t_2\ - t_1| <\ \epsilon, v \in R, i = 1,2\ \right\}.$$

Then, form estimate we get

$$|T_1y(t_2) - T_1y(t_1)| \le\ \aleph(h_1,\epsilon) + [m_1(t_2)\aleph(y,\epsilon)\aleph_{fi}(\epsilon)]n_1\varphi_1(\||\ y|\|) + \int_0^{t_2} d_s\left(\vee_{p=0}^{s} g_1(t_2,p)\right)$$

$$+\ [m_1\ \|y\| + F_1]\left[n_1\varphi_1(\||\ y|\|) \int_0^{t_2} d_s\left(\vee_{p=0}^{s}[g_1(t_2,p) - g_1(t_1,p)]\right)\right.$$

$$\left. + \aleph_{u1}(\epsilon) \int_0^{t_2} d_s\left(\vee_{p=0}^{s} g_1(t_2,p)\right) + n_1\varphi_1(\||\ y|\|) \int_{t_1}^{t_2} d_s\left(\vee_{p=0}^{s} g_1(t_1,p)\right)\right]$$

$$\le\ \aleph(h_1,\epsilon) + [m_1(t_2)\aleph(y,\epsilon)\aleph_{fi}(\epsilon)]n_1\varphi_1(\||\ y|\|) + \int_0^{t_2} d_s(\vee_{s=0}^{t} g_1(t_2,s))$$

$$+\ [m_1\ \|y\| + F_1]\left[n_1\varphi_1(\||\ y|\|) \int_0^{t_2} d_s(\vee_{s=0}^{t}[g_1(t_2,s) - g_1(t_1,s)])\right.$$

$$\left. + \aleph_{u1}(\epsilon) \int_0^{t_2} d_s(\vee_{s=0}^{s} g_1(t_2,s)) + n_1\varphi_1(\||\ y|\|) \int_{t_1}^{t_2} d_s(\vee_{s=0}^{t} g_1(t_1,s))\right]$$

$$\le\ \aleph(h_1,\epsilon) + K_1[m_1\aleph(y,\epsilon) + \aleph_{f1}(\epsilon)]n_1\varphi_1(r)$$

$$+ [m_1\, r + F_1]\big[n_1\varphi_1(r)W_1(\epsilon) + \aleph_{f1}(\epsilon)[g_1(t_1,t_2) - g_1(t_1,0)]$$

$$+ n_1\varphi_1(r)[g_1(t_1,t_2) - g_1(t_1,t_1)]\big].$$

Hence, from the continuity of the functions g_1 assumption (vi), we deduce that T_1 maps $C(I)$ into $C(I)$. As done above we can obtain

$$|T_2y(t_2) - T_2y(t_1)| \le \aleph(h_2,\epsilon) + K_2[m_2\aleph(y,\epsilon) + \aleph_{f2}(\epsilon)]n_2\varphi_2(r)$$

$$+ [m_2\, r + F_2]\big[n_2\varphi_2(r)W_2(\epsilon) + \aleph_{f2}(\epsilon)[g_2(t_1,t_2) - g_2(t_1,0)]$$

$$+ n_2\varphi_2(r)[g_2(t_1,t_2) - g_2(t_1,t_1)]\big]$$

Also, by our assumption (iv), we see that T_2 maps $C(I)$ into $C(I)$.
Now, from the definition of the operator T we get

$$Tv(t_2) - Tv(t_1) = T(x,y)(t_2) - T(x,y)(t_1)$$

$$= \big(T_1y(t_2), T_2x(t_2)\big) - (T_1y(t_1), T_2x(t_1))$$

$$= \big(T_1y(t_2) - T_1y(t_1), T_2x(t_2) - T_2x(t_1)\big).$$

Therefore

$$\|Tv(t_2) - Tv(t_1)\| = \big\|\big(T_1y(t_2) - T_1y(t_1), T_2x(t_2) - T_2x(t_1)\big)\big\|$$

$$= max\,\{\|T_1y(t_2) - T_1y(t_1)\|,\ \|T_2y(t_2) - T_2y(t_1)\|\}$$

$$\le max\,\big\{\aleph(h_1,\epsilon) + K_1[m_1\aleph(y,\epsilon) + \aleph_{f1}(\epsilon)]n_1\varphi_1(r)$$

$$+ [m_1 r + F_1\}[n_1\varphi_1(r)W_1(\epsilon) + \aleph_{u1}(\epsilon)[g_1(t_1,t_2) - g_1(t_1,0)]$$

$$+ n_1\varphi_1(r)[g_1(t_1,t_2) - g_1(t_1,t_1)]],$$

$$\aleph(h_2,\epsilon) + K_2[m_2\aleph(y,\epsilon) + \aleph_{f2}(\epsilon)]n_2\varphi_2(r)$$

$$+ [m_2 r + F_2\}[n_2\varphi_2(r)W_2(\epsilon) + \aleph_{u2}(\epsilon)[g_2(t_1,t_2) - g_2(t_1,0)]$$

$$+ n_2\varphi_2(r)[g_2(t_1,t_2) - g_2(t_1,t_1)]]\}.$$

This means that the class of $\{Tv(t)\}$ is equi-continuous on I, then by Arzela-Ascoil theorem $\{Tv(t)\}$ is relatively compact.
Now, we will show that the operator $T : V \to V$ is continuous.
Firstly, we prove that T_1 is continuous. Let$\epsilon * > 0$, the continuity of$u_i, i = 1,2$, yields, $\exists\ \delta = \delta(\epsilon *)$ such that $\left|u_i(t,s,u(s)) - u_i(t,s,v(s))\right| < \epsilon *$, whenever $\| u - v \| \leq \delta$, thus if $\| y - u \| \leq \delta$, we arrive at:

$$|(T_1 y)(t) - (T_1 u)(t)|$$

$$\leq \Big| f_1(t,y(\omega_1(t)) \int_0^t u_1(t,s,y(\omega_1(s))d_s g_1(t,s)$$

$$-f_1(t,u(\omega_1(t)) \int_0^t u_1(t,s,u(\omega_1(s))d_s g_1(t,s)\Big|$$

$$\leq \Big| f_1(t,y(\omega_1(t)) \int_0^t u_1(t,s,u(\omega_1(s))d_s g_1(t,s)$$

$$- f_1(t,u(\omega_1(t)) \int_0^t u_1(t,s,y(\omega_1(s))d_s g_1(t,s)\Big|$$

$$+ f_1(t,u(\omega_1(t)) \int_0^t u_1(t,s,y(\omega_1(s))d_s g_1(t,s)$$

$$- f_1(t,u(\omega_1(t)) \int_0^t u_1(t,s,u(\omega_1(s))d_s g_1(t,s)\Big|$$

$$\leq \left|f_1(t,y(\omega_1(t)) - f_1(t,u(\omega_1(t))\right| \int_0^t \left|u_1(t,s,y(\omega_1(s))\right| d_s g_1(t,s)$$

$$+ |f_1(t,u(\omega_1(t))| \int_0^t \left|u_1(t,s,y(\omega_1(s)) - u_1(t,s,u(\omega_1(s))\right| d_s g_1(t,s)$$

$$\leq \; m_1(t)|y(t)-u(t)|\int_0^t n_1(t,s)\varphi_1(|y(s)|)\; d_s g_1(t,s)$$
$$+[m_1\,(t)|u(t_1)|+$$
$$|f_1(t_1,0)|]\int_0^t \left|u_1(t,s,y\big(\,\omega_1(s)\big)-u_1(t,s,u\big(\,\omega_1(s)\big)\right|\,d_s g_1(t,s)$$

$$\leq (\delta m_1 n_1 \varphi_1(\|y\|)+[m_1\|u\|+F_1]\epsilon^*)\int_0^t d_s \bigvee_{p=0}^{s} g_1(t,p)$$

$$\leq (\delta m_1 n_1 \varphi_1(\|y\|)+[m_1\|u\|+F_1]\epsilon^*)\bigvee_{s=0}^{t} g_1(t,s),$$

$$\leq (\delta m_1 n_1 \varphi_1(\|y\|)+[m_1\|u\|+F_1]\epsilon^*)K_1,$$

where

$$\epsilon = (\delta m_1 n_1 \varphi_1(\|y\|)+[m_1\|u\|+F_1]\epsilon^*)K_1.$$

Therefore

$$|(T_1 y)(t)-(T_1 u)(t)| \leq \epsilon.$$

This means that the operator T_1 is continuous.
By a similar way as done above we can prove that for any $x, u \in C[0,T]$ and $\| x - v \| < \delta$, we have

$$|(T_2 x)(t)-(T_2 v)(t)| \leq \epsilon.$$

Hence T_2 is continuous operator. The operators T_i $(i = 1, 2)$ are continuous operator this imply that T is continuous operator. Since all conditions of Schauder fixed point theorem are satisfied, then T has at least one fixed point $v = (x, y) \in V$, which completes the proof

Corollary 1. Let assumptions of Theorem 1 be satisfied. Then quadratic Volterra-Stieltjes functional integral equation

$$x(t) = \; h(t)+f(t,y\big(\omega(t)\big)\int_0^t u(t,s,y\big(\omega(s)\big)d_s g(t,s)\,,$$
$t \in I$ **(3)**

has at least one solution $x \in C(I)$.

Proof. Let the assumptions of Theorem 1 be satisfied. With $x = y$, $f_1 = f_2 = f$, $u_1 = u_2 = u$, and $h_1 = h_2 = h$ **.** Then the coupled system (1) will be the Volterra-Stieltjes quadratic integral equation (3)

3. Existence of unique solution

Here, we study the uniqueness of the solution $(x, y) \in X$ of the coupled system of
quadratic Volterra-Stieltjes integral equations (1). Assume that functions $\varphi_i : R+ \to R+$ have the
form $\varphi_i(x) = 1 + |x|$, and the functions $n_i(t, s) \in C(I)$ denoted by $b_i = \| n_i \| = max\{n_i(t, s)\ t, s \in I, i = 1, 2\}$. Then

$$|u_i(t, s, x)| \leq n_i(t, s)(1 + |x|).$$

Notice that this assumption is a special case of assumption (iii).
Consider now the assumptions (ii)*, (iii)* having the form

(ii)* $f_i: I \times R \to R$ are continuous functions and there exist constants numbers m_i such that

$$|f_i(t, x) - f_i(t, y)| \leq m_i|x - y|, \qquad i = 1, 2.$$

From this assumption, we can deduce that

$$|u_i(t, s, x)| - |u_i(t, s, 0)| \leq |u_i(t, s, x) - u_i(t, s, 0)| \leq b_i|x|,$$

which implies that

$$|u_i(t, s, x)| \leq |u_i(t, s, 0)| + b_i|x| \leq n_i(t, s) + b_i|x|,$$

where $n_i(t, s) = \sup_{t \in I} |u_i(t, s, 0)|$.

Theorem 2. Let assumptions of Theorem 1 be satisfied with replace assumptions (ii), (iii) by (ii)*, (iii)*, if the following conditions hold

$$m(n + br) + (mr + F)K \leq 1.$$

Then the coupled system (1) has an unique solution $(x, y) \in X$.

Proof. Let $\boldsymbol{v_1} = (\boldsymbol{x_1}, \boldsymbol{y_1})$ and $\boldsymbol{v_2} = (\boldsymbol{x_2}, \boldsymbol{y_2})$ be two solutions of the coupled system (1), we have

$$\|(\boldsymbol{x_1}, \boldsymbol{y_1}) - (\boldsymbol{x_2}, \boldsymbol{y_2})\|_X = \|(\boldsymbol{x_1} - \boldsymbol{x_2}, \boldsymbol{y_1} - \boldsymbol{y_2}\|_X$$

$$= \max_{t\in I}\{\|\boldsymbol{x_1} - \boldsymbol{x_2}\|, \|\boldsymbol{y_1} - \boldsymbol{y_2}\|\}.$$

Now

$$|\boldsymbol{x_1(t)} - \boldsymbol{x_2(t)}|$$

$$\leq \left|\boldsymbol{f_1(t, y_1(\omega_1(t))} \int_0^t \boldsymbol{u_1(t, s, y_1(\omega_1(s))d_s g_1(t, s)}\right.$$

$$\left. - \boldsymbol{f_1(t, y_2(\omega_1(t))} \int_0^t \boldsymbol{u_1(t, s, y_2(\omega_1(s))d_s g_1(t, s)}\right|$$

$$\leq \left|\boldsymbol{f_1(t, y_1(\omega_1(t))} \int_0^t \boldsymbol{u_1(t, s, y_1(\omega_1(s))d_s g_1(t, s)}\right.$$

$$\left. - \boldsymbol{f_1(t, y_2(\omega_1(t))} \int_0^t \boldsymbol{u_1(t, s, y_1(\omega_1(s))d_s g_1(t, s)}\right|$$

$$+ \left|\boldsymbol{f_1(t, y_2(\omega_1(t))} \int_0^t \boldsymbol{u_1(t, s, y_1(\omega_1(s))d_s g_1(t, s)}\right.$$

$$\left. - \boldsymbol{f_1(t, y_2(\omega_1(t))} \int_0^t \boldsymbol{u_1(t, s, y_2(\omega_1(s))d_s g_1(t, s)}\right|$$

$$\leq \left|\boldsymbol{f_1(t, y_1(\omega_1(t)) - f_1(t, y_2(\omega_1(t))}\right| \int_0^t \left|\boldsymbol{u_1(t, s, y_1(\omega_1(s))}\right| \boldsymbol{d_s g_1(t, s)}$$

$$+ |\boldsymbol{f_1(t, y_2(\omega_1(t))}| \int_0^t \left|\boldsymbol{u_1(t, s, y_1(\omega_1(s)) -}\right.$$
$$\left.\boldsymbol{u_1(t, s, y_2(\omega_1(s))}\right| \boldsymbol{d_s g_1(t, s)}$$

$$\leq \boldsymbol{m_1|y_1(t) - y_2(t)|} \int_0^t \boldsymbol{(n_1(t, s) + b_1|y|)d_s g_1(t, s)}$$

$$+ [\boldsymbol{m_1|y_2(t)| + |f_1(t, 0)|}]\boldsymbol{b_i} \int_0^t \boldsymbol{|y_1(t) - y_2(t)|\, d_s g_1(t, s)}$$

$$\leq [\|\boldsymbol{y_1 - y_2}\|\boldsymbol{m_1(n_1 - b_1\|y_1\|)} + [\boldsymbol{m_1\|y_2\| + F_1}]\|\boldsymbol{y_1 -}$$
$$\boldsymbol{y_2}\| \int_0^t \boldsymbol{d_s}(\vee_{\boldsymbol{p=0}}^{\boldsymbol{s}} \boldsymbol{g_1(t, p)})$$

$$\leq \boldsymbol{m_1(n_1 + b_1\|y_1\|)} + [\boldsymbol{m_1\|y_2\| + F_1\|y_1 - y_2\|}](\vee_{\boldsymbol{s=0}}^{\boldsymbol{t}} \boldsymbol{g_1(t, s)})$$

$$\leq \boldsymbol{m_1(n_1 + b_1 r_1) + [m_1 r_1 + F_1]K_1\|y_1 - y_2\|}.$$

Therefore

$$\|x_1 - x_2\| \leq m(n + br) + [mr + F]K\|y_1 - y_2\|,$$

where

$b = max\{b_1, b_2\}$, $m = max\{m_1, m_2\}$, $n = max\{n_1, n_2\}$, $F = max\{F_1, F_2\}$ and $K = max\,\{K_1, K_2\}$.

Similarly

$$\|y_1 - y_2\| \leq m(n + br) + [mr + F]K\|x_1 - x_2\|.$$

Then

$$\|(x_1, y_1) - (x_2, y_2)\|_X = \|(x_1 - x_2, y_1 - y_2\|_X$$

$$= \max_{t\in I}\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}$$

$$= \max_{t\in I}\{m(n + br) + [mr + F]K\|y_1 - y_2\| m(n + br) + [mr + F]K\|x_1 - x_2\|\}$$

$$= [m(n + br) + [mr + F]K] \max_{t\in I}\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}$$

$$= m(n + br) + [mr + F]K\;\|(x_1, y_1) - (x_2, y_2)\|_X.$$

Which implies that

$$[1 - m(n + br) + [mr + F]K]\|(x_1, y_1) - (x_2, y_2)\|_X \leq 0.$$

This means that

$$(x_1, y_1) = (x_2, y_2) \implies x_1 = x_2, \qquad y_1 = y_2.$$

This proves the uniqueness of the solution of the coupled system (1).

4. SPECIAL CASES

In this section, we will consider a coupled system of quadratic Volterraintegral equations of fractional order, which has form

$$x(t) = h_1(t) + f_1(t, y(\omega_1(t))) \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha_1)}\, u_1(t, s, y(\omega_1(t)))\, ds,\; t \in I$$

(4)

$$y(t) = h_1(t) + f_2(t, x(\omega_2(t))) \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha_1)} u_2(t, s, x(\omega_2(t)))\, ds, \ t \in I$$

where $t \in I = [0, T]$ and $\alpha_i \in (0,1)$, and $\Gamma(\alpha_i), i = 1, 2$, refers to gamma functions. Let us mention that (4) represents the so-called a coupled systems of Volterra quadratic integral equations of fractional order. Recently, such a type this type has been widely investigated in some papers [1, 9, 10, 12, 13, 14]
Here, we show that a coupled systems of fractional orders (4) can be treated as a special case of a coupled systems of quadratic Volterra-Stieltjes integral equations (2) studied in Section 2.
In fact, we can consider functions $g_i(t,s) = g_i : \triangle \rightarrow R, i = 1, 2$, defined by the formula

$$g_i(t,s) = \frac{t^{\alpha_i} - (t-s)^{\alpha_i}}{\Gamma(\alpha_i + 1)}.$$

We can see that functions $g_i, i = 1, 2$, satisfy assumptions (vi)-(vii) in Theorem 1, see [6, 8].
Now, we state the following existence results for couple system of quadratic Volterra integral equations of fractional order (4).

Theorem 3. Let assumptions (i)-(viii) of Theorem 1 be satisfied. Then a coupled systems of fractional orders (4) has at least one solution $(x, y) \in X$.

Corollary 2. Let assumptions of Theorem 3 be satisfied (with $= y, u_1 = u_2 = u$ $, f_1 = f_2 = f, h_1 = h_2 = h$ and $\alpha_1 = \alpha_2 = \alpha$**).** Then the fractional-order quadratic integral equation

$$x(t) = h(t) + f(t, x(\omega(t))) \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(t, s, x(\omega(t)))\, ds, \ t \in I$$

has at least one solution in x ∈ C(I) .

Corollary 3. Let assumptions of Theorem 3 be satisfied, with $f_1(t, y(t)) = f_2(t, x(t)) = 1$. Then a coupled system of the fractional-order quadratic integral system

$$x(t) = h_1(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_1(t, s, y(\omega_1(t)))\, ds, \qquad t \in I$$

(5)

$$y(t) = h_1(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} u_2(t, s, x(\omega_2(t)))\, ds, \qquad t \in I$$

has at least one solution in $(x, y) \in X$.

Now, letting $\alpha_1, \alpha_2 \to 1$, we obtain

Corollary 4. Let assumptions of Theorem 3 be satisfied. Then the coupled system of the initial value problems

$$\frac{x(t)}{dt} = u_1\left(t, s, y\left(\omega_1(t)\right)\right), \qquad t \in I, \quad x(0) = x_0,$$

(6)

$$\frac{y(t)}{dt} = u_1\left(t, s, x\left(\omega_2(t)\right)\right), \qquad t \in I, \quad y(0) = y_0,$$

Proof. Let assumptions of Theorem 3 be satisfied (with $f_1\big(t, y(t)\big) = f_2\big(t, x(t)\big) = 1, h_1(t) = x_0,\ h_2(t) = y_0$ and letting $\alpha, \beta \to 1$. Then a coupled system of the fractional-order quadratic integral equations

$$x(t) = x_0 + \int_0^t u_1(t, s, y(\omega_1(t)))\, ds, \qquad t \in I,$$

(7)

$$y(t) = y_0 + \int_0^t u_1(t, s, x(\omega_2(t)))\, ds, \qquad t \in I,$$

has at least one solution in X which is equivalent to the coupled system of the initial value problems (6).

Corollary 5. Let assumptions of Theorem 3 be satisfied. Then the coupled system of fractional order differential equations

$$D^{\alpha_1}x(t) = u_1\left(t, s, y\left(\omega_1(t)\right)\right), \qquad t \in I$$

(8)

$$D^{\alpha_2}y(t) = u_2\left(t, s, x\left(\omega_2(t)\right)\right), \qquad t \in I,$$

with the initial conditions

$$I^{1-\alpha_1}x(t)\,\big|_{t=0} = I^{1-\alpha_2}y(t)\,\big|_{t=0} = 0,\ \ \alpha_1, \alpha_2 \in (0, 1],$$

(9)

has at least one solution in $(x, y) \in X$.

Proof. let us proof the coupled system of the initial value problems (8) and (9) is equivalent to the coupled system of quadratic integral system

$$x(t) = \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, u_1(t, s, y(\omega_1(t)))\, ds, \qquad t \in I$$

(10)

$$y(t) = \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)}\, u_2(t, s, x(\omega_2(t)))\, ds, \qquad t \in I$$

By operating $I^{1-\alpha_1}$ and $I^{1-\alpha_2}$respectively on each equation of coupled system (10), and applying properties of fractional operator [23], we obtain

$$I^{1-\alpha_1}x(t) = I^1 u_1(t, s, y(\omega_1(t))), \qquad I^{1-\alpha_1}x(t)\,\big|_{t=0} = 0$$

$$I^{1-\alpha_2}y(t) = I^1 u_2(t, s, y(\omega_2(t))), \qquad I^{1-\alpha_2}y(t)\,\big|_{t=0} = 0.$$

Also,

$$\frac{d}{dt} I^{1-\alpha_1}x(t) = u_1\left(t, s, y\left(\omega_1(t)\right)\right),\ \ t \in I,\ \ \alpha_1 \in (0, 1)$$

$$\frac{d}{dt} I^{1-\alpha_2} y(t) = u_2\left(t, s, y\left(\omega_2(t)\right)\right), \; t \in I, \; \alpha_2 \in (0, 1).$$

Conversely, by integrating the coupled system of initial value problems (8) and (9), we have

$$I^{1-\alpha_1} x(t) - I^{1-\alpha_1} x(t)\Big|_{t=0} = I^1 u_1(t, s, y(\omega_1(t)))$$

$$I^{1-\alpha_2} y(t) - I^{1-\alpha_2} y(t)\Big|_{t=0} = I^1 u_2\left(t, s, y\left(\omega_2(t)\right)\right).$$

Operating by I^{α_1} and I^{α_2} respectively on each equation and differentiating, we have (10). Thus, the equivalence hold.

Let assumptions of Theorem 3 be satisfied (with $f_1\left(t, y\left(\omega_1(t)\right)\right) = f_2\left(t, x\left(\omega_2(t)\right)\right) = 1, h_1(t) = h_2(t) = 0$. Then there exists at least one solution in X for the coupled system (8 and 9).

References

[1] Sh. M Al-Issa, A. M. A. El-Sayed, *Positive integrable solutions for nonlinear integral and differential inclusions of fractional-orders*, Comment. Math. 49(2009), no.2, 157–159.

[2] M. M. A. Al-Fadel,, *On a coupled system of Volterra-Stieltjes integral equations*, Electron. J. Math. Anal. Appl.7(2019), no. 1, 95–101.

[3] M. M. A. Alfadel, A. M. A. El-sayed, A. (2017). On the weak solutions of a coupled system of Volterra-Stieltjes integral equations. Comment. Math., 57(2017), no,2, 143–151.

[4] J. Bana´s, and J. C. Mena, *Some properties of nonlinear Volterra-Stieltjes integral operators*, Comput. Math. Appl., 49(2005) no. 9-10, 1565–1573.

[5] J. Bana´s, *Existence results for Volterra-Stieltjes quadratic integral equations on an unbounded interval*, Math. Scand., (2006), 143–160.

[6] J. Bana´s, T. Zajac, T*A new approach to the theory of functional integral equations of fractional order*. J. Math. Anal. Appl. , 375(2011), no.2, 375-387

[7] J. Bana´s, D. Agnieszka, *Solutions of a quadratic Volterra-Stieltjes integral equation in the class of functions converging at infinity*, Electron. J. Qual. Theory Differ. Equ., 80(2018), 1-17.

[8] A.M.A. El-Sayed and W.G. Ibrahim and A.A.H. Mowla, *Weak solutions of fractional order differential equations via Volterra-Stieltjes integral operator*, J. Math. Appl., 40 (2017), 85–96.

[9] A. M. A. El-Sayed, Sh. M Al-Issa, *Global Integrable Solution for a Nonlinear Functional Integral Inclusion*, SRX Mathematics. 2010 (2010).

[10] A. M. A. El-Sayed, Sh. M Al-Issa, *Monotonic continuous solution for a mixed type integral inclusion of fractional order*, J. Math. Appl., 33 (2010),27–34.
[11] A. M. A. El-Sayed, Sh. M. Al-Issa, *Monotonic solutions for a quadratic integral equation of fractional order*, AIMS Math., 4 (2019), no. 3, 821–830.
[12] A. M. A. El-Sayed, Sh. M Al-Issa, *Monotonic integrable solution for a mixed type integral and differential inclusion of fractional orders*, Int. J. Differ. Equations Appl., 18 (2019), no.1. 1–9.
[13] A. M. A. El-Sayed, Sh. M Al-Issa, *Existence of continuous solutions for nonlinear functional differential and integral inclusions*, Malaya Journal of Matematik, 7(3) (2019), 545–549.
[14] A. M. A. El-Sayed, Sh. M Al-Issa, *Existence of integrable solutions for integro-differential inclusions of fractional order; coupled system approach*, J. Nonlinear Sci. Appl., 13(2020), no. 4, 180–186.
[15] A. M. A. El-Sayed, Sh. M Al-Issa, *On the existence of solutions of a set-valued functional integral equation of Volterra-Stieltjes type and some applications*. Adv. Difference Equ. 2020 (2020), Paper No. 59, 16 pp.
[16] A. M. A. El-Sayed and Sh. M Al-Issa, *On set-valued functional integral equations of Hammerstein-Stieltjes type: existence of solutions, continuous dependence, and applications, Methods Funct. Anal. Topology*, 27 (2021), no.
2, 157–172.
[17] A. M. A. El-Sayed, Sh. M Al-Issa, *On a set-valued functional integral equation of Volterra-Stieltjes type*, J.Math. Computer Sci., 21 (2020), no. 4, 273–285. 236 SH. M AL-ISSA JFCA-2022/13(2)
[18] A.M.A. El-Sayed, Sh.M Al-Issa, *On set-valued functional integral equations of Hammerstein-Stieltjes type: existence of solutions, continuous dependence, and applications*, Methods Funct. Anal. Topology, 27 (2021), no.
2, 157–172.
[19] A. M. A. El-Sayed, Sh. M Al-Issa, and Y. Omar, *On Chandrasekhar functional integral inclusion and Chandrasekhar quadratic integral equation via a nonlinear UrysohnStieltjes functional integral inclusion*, Adv Differ
Equ 2021 (2021), no.137, 17 pages.
[20] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, Inc., New York, 1993.
[21] S. G. Samko, A. A. Kilbas and O. Marichev, *Integrals and derivatives of fractional order and some of their applications*, Nauka i Teknika, Minsk, 1987.
[22] I. Podlubny and A. M. A. EL-Sayed, *On two defintions of fractional calculus*, Solvak Academy of science-Institute of Experimental phys. 1996.
[23] I. Podlubny, *Fractional differential equation*, Acad. Press, San Diego-New York-london, 1999.