



## Analysis Of A Coupled Quadratic Volterra-Stieltjes Integral Equation System

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#### **Abstract:**

This paper aims to establish existence results for a coupled system of nonlinear quadratic integral equations of the Volterra-Stieltjes type in the space of continuous functions over a closed bounded interval. The existence of solutions is demonstrated using the Schauder fixed point principle. This approach enables us to derive existence theorems under broad and general conditions.

**Key words:** Coupled system, nonlinear quadratic integral equation, function of bounded variation, fractional order

#### 1. Introduction

Integral equations play a crucical role in modeling various phenomena and events in applied sciences. This field has witnessed significant development through the use of functional analysis, topology, and fixed point theory (see[1, 9, 10, 11, 12, 15]). Among these advancements, Volterra-Stieltjes integral equations have attracted considerable attention, with several studies dedicated to their analysis (see [5, 7, 16, 17, 18, 19]).

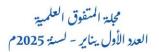
The primary objective of this paper is to examine the solvability of a coupled system of quadratic Volterra-Stieltjes integral equations.

Give the importance of the Stieltjes integral in this context; we rely on the definitions and properties introduced by Banas` (see [2, 3]).

Additionally, interest in studying such coupled system has been sparked by previous research on related topics.

In this study, we establish some existence theorems for a coupled system of quadratic Volterra-Stieltjes integral equations, which encompass various typs of Volterra integral equations as special cases. Our proof is based on the fixed-point principle, enabling us to derive existence results under broad and flexible assumptions.

Throughout this paper, let I = [0, T] and X be the Banach space of all ordered pairs  $(x, y) \in X = C(I) \times C(I)$ , with the norm





$$||(x,y)||_X = max\{||x||_C, ||y||_C\},$$

where

$$||x||_C = \sup_{t \in I} |x|, ||y||_C = \sup_{t \in I} |y|,$$

It is clear that  $(X, ||(x, y)||_X)$  is Banach space.

Now, we shall present some auxiliary properties of fractional calculus that will be need in this work.

**Definition 1.** The Riemann-Liouville of a fractional integral of the function  $f \in L_1(I)$  of order  $\alpha \in R^+$  is defined by

$$I_a^{\alpha}f(t)=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\ ds.$$

and when h = 0, we have  $I^{\alpha} f(t) = I_a^{\alpha} f(t)$ .

**Definition 2**. The (Caputo) fractional order derivative  $D^{\alpha}$ ,  $\alpha \in (0,1]$  of the absolutely continuous function g is defined as

$$D_a^{\alpha}g(t)=I_a^{1-\alpha}\frac{d}{dt}g(t), \qquad t\in[a,b].$$

For further properties of fractional calculus operator (See [20], [21], [22] and [23]).

#### 2. Preliminaries

In this section, we study the solvability of the coupled system of nonlinear quadratic integral equations of Volterra-Stieltjes type having the form

$$x(t) = h_1(t) + f_1(t, y(\omega_1(t))) \int_0^t u_1(t, s, y(\omega_1(s))) d_s g_1(t, s), \quad t \in I$$

**(1)** 

$$y(t) = h_2(t) + f_{12}(t, y(\omega_2(t))) \int_0^t u_2(t, s, y(\omega_2(s))) d_s g_2(t, s), \quad t \in I$$



Our goal is to show that system (1) has at least one solution in the space X. For our further purposes we denote by  $\triangle$  the triangle  $\triangle = \{(t, s): 0 \le s \le t \le T\}$ .

In our investigations, we give some assumptions which are needed throughout this paper.

- (i)  $h_i$ :  $I \to R$ , (i = 1, 2) are continuous on I. There are constants  $h_i$ , where  $h_i = \sup_{t \in I} |h_i(t)|$ .
- (ii)  $fi: I \times R \rightarrow R, (i = 1, 2)$  are continuous functions and there exist continuous functions

$$m_i(t): I \rightarrow I$$
 such that

$$|f_i(t,x) - f_i(t,y)| \leq m_i(t)|x - y|,$$

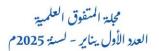
for all  $x, y \in R$  and  $t \in I$ . Moreover, we put  $m_i = max\{m_i(t) \mid t \in I, \}$ .

(iii)  $u_i(t,s,x): \triangle \times R \to R$ , (i=1,2) are continuous functions and there exist continuous functions  $n_i(t,s): \triangle \to I$ , and continuous and nondecreasing functions  $\varphi_i: R+\to R+$ , such that

$$|u_i(t,s,x)| \leq n_i(t,s)\varphi_i(|x|),$$

for all  $(t,s) \in \Delta$  and  $x \in R$ . Moreover, we put  $n_i = \max\{n_i(t,s) \ t,s \in I\}$ .

- (iv)  $\omega_i : I \to I$  are continuous, (i = 1, 2).
- (v) Functions  $s \to g_i(t, s)$  are of bounded variation on [0, t] for each  $t \in I$ , i = 1, 2.
- (vi) Functions  $g_i$  are continuous on the triangle  $\triangle$  and  $g_i(t,0) = 0$  for i = 1, 2.
- (vii)  $g_i(t,s) = g_i : \Delta_i \to R, i = 1,2$  and for all  $t_1, t_2 \in I$  with  $t_1 < t_2$ , the functions  $s \to g_i(t_2,s) g_i(t_1,s)$  are nondecreasing on I.
- (viii) For any  $\epsilon>0$  there exists  $\delta>0$  such that, for all  $t_1$ ;  $t_2\in I$  such that  $t_1< t_2$  and  $t_2-t_1\leq \delta$





the following inequality holds

$$\bigvee_{s=0}^{t} [g_i(t_2, s) - g_i(t_1, s)] \le \epsilon, \ i = 1, 2.$$

Obviously, we will assume that  $g_i$ , (i = 1, 2) satisfy assumptions (iv) - (vii). For our purposes, we will need the following lemmas.

**Lemma 1.** [6] Under assumptions (v)-(viii), The functions  $z \to \bigvee_{s=0}^{z} g_i(t,s)$  are continuous on [0,t] for any  $t \in I$  (i=1,2).

**Lemma 2.** [6] Let assumptions (v)-(viii) be satisfied. Then, for arbitrary fixed number  $0 < t_2 \in I$ 

and for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $t_1 \in I$ ;  $t_1 < t_2$  and  $t_2 - t_1 \leq \delta$  then  $\bigvee_{s=t_1}^{t_2} g_i(t_2,s) \leq \epsilon$ . (i=1,2).

Further, let us observe that based on Lemma 1 we infer that there exists finite positive constants

 $K_i$ , such that

$$K_i = \sup \left\{ \bigvee_{s=0}^t g_i(t,s) : t \in [0,T] \right\}.$$

where T > 0 is arbitrarily fixed (i = 1, 2).

We now introduce some functions that will be useful in our further studies:

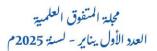
$$W_{i} = \sup \left\{ \bigvee_{s=0}^{t_{2}} \left( g_{i}(t_{2}, s) - g_{i}(t_{1}, s) \right) : t_{1}, t_{1} \in I, t_{1} < t_{2} \leq \epsilon, i = 1, 2 \right\}.$$

In what follows let us denote by  $F_i$  the constant defined by the formula:

$$F_i = \sup\{|f_i(t,0)|: t \in I, i = 1, 2\}.$$

Now, we are in position to present tha main result of the paper.

#### 3. Main Result





Defined the operator by

$$T(x,y)(t) = (T_1y(t), T_2x(t)),$$

where

$$T_1x(t) = h_1(t) + f_1(t, y(\omega_1(t))) \int_0^t u_1(s, y(\omega_1(s))) d_s g_1(t, s), \ t \in I$$
(2)

$$T_2 y(t) = h_2(t) + f_{12}(t, y(\omega_2(t))) \int_0^t u_2(s, y(\omega_2(s))) d_s g_2(t, s), \qquad t \in I$$

**Theorem 1.** Let assumptions (i)-(vii) be satisfied. Then the coupled system of quadratic Volterra-Stieltjes integral equations (1) has at least one solution (x, y) belonging to the space X.

**Proof**. We prove a few results concerning the continuity and compactness of these operators in the space of continuous functions.

Define

$$V = \{v = (x(t), y(t)) : (x(t), y(t)) \in X : ||(x, y)||_X \le r\}.$$

For  $(x, y) \in V$ , and define the operator T map V into V, we have

$$\begin{split} |T_1y(t)| &\leq |h_1| + |f_1(t,y(\omega_1(t))| \int_0^t |u_1(t,s,y(\omega_1(s)))| |d_sg_1(t,s)| \\ &\leq \|h_1\| + [m_i|y(t)| + \\ |f_1(t,0)|] \int_0^t n_1(t,s)\varphi_1(|y(s)|) d_s(\bigvee_{P=0}^t g_1(t,P), \\ &\leq \|h_1\| + [\|y\|m_1 + F_1] n_1\varphi_1(\|y\|) (\bigvee_{p=0}^t g_1(t,p), \\ \|T_1y\| &\leq \|h_1\| + [r_1m_1 + F_1] n_1\varphi_1(r_1) \sup_{t \in I} (\bigvee_{p=0}^t g_1(t,p)). \end{split}$$

Hence, we get

$$||T_1y|| \leq ||h_1|| + K_1[m_1r_1 + F_1]n_1\varphi_1(r_1).$$

From the last estimate we deduce that  $r_1 = \frac{\|h_1\| + F_1 K_1 n_1 \varphi_1(r_1)}{1 - m_1 n_1 K_1 \varphi_1(r_1)}$ .

By a similar way, we obtain



$$||T_2y|| \le ||h_2|| + K_2[m_2r_2 + F_2]n_2\varphi_2(r_2).r_2 = \frac{||h_2|| + F_2K_2n_2\varphi_2(r_2)}{1 - m_2n_2K_2\varphi_2(r_2)}.$$

Therefore

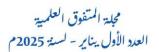
$$||Tv||_{X} = ||T(x,y)||_{X} = ||T_{1}y,T_{2}x||_{X}$$

$$\leq \max_{t\in I} \{||T_{1}y||_{C},||T_{2}y||_{C}\} = r.$$

Thus for every  $v = (x, y) \in V$ , we have  $Tv \in V$  and hence  $TV \subset V$ ,  $(i.eT: V \to V)$ . This means that the functions of TU are uniformly bounded on I, it is clear that the set V is nonempty, bounded, closed and convex. Now, we need to show that the set TV is relatively compact.

For  $v = (x, y) \in V$ , for all  $\epsilon > 0$ ,  $\delta > 0$ , and for each  $t_1, t_2 \in I$  (without loss of generality assume that  $t_1 < t_2$ , such that  $|t_2 - t_1| < \delta$ , we have

$$\begin{split} &|T_{1}y(t_{2})-T_{1}y(t_{1})|\\ &=\Big|h_{1}(t_{2})-h_{1}(t_{1})+\\ &f_{1}(t_{2},y(\omega_{1}(t_{2})))\int_{0}^{t_{2}}u_{1}\Big(t_{2},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{2},s)\\ &-f_{1}(t_{1},y(\omega_{1}(t_{1})))\int_{0}^{t_{1}}u_{1}\Big(t_{1},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{1},s)\Big|\\ &\leq|h_{1}(t_{2})-h_{1}(t_{1})|+|f_{1}(t_{2},y(\omega_{1}(t_{2})))-f_{1}(t_{2},y(\omega_{1}(t_{1})))|\\ &\cdot\Big|\int_{0}^{t_{2}}u_{1}\Big(t_{2},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{2},s)\Big|\\ &+|f_{1}(t_{2},y(\omega_{1}(t_{1})))\int_{0}^{t_{2}}u_{1}\Big(t_{2},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{2},s)\Big|\\ &-|f_{1}(t_{1},y(\omega_{1}(t_{1})))\int_{0}^{t_{2}}u_{1}\Big(t_{2},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{2},s)\Big|\\ &+|f_{1}(t_{1},y(\omega_{1}(t_{1})))\int_{0}^{t_{2}}u_{1}\Big(t_{2},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{2},s)\Big|\\ &-|f_{1}(t_{1},y(\omega_{1}(t_{1})))\int_{0}^{t_{2}}u_{1}\Big(t_{2},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{2},s)\Big|\\ &-|f_{1}(t_{1},y(\omega_{1}(t_{1})))\int_{0}^{t_{2}}u_{1}\Big(t_{2},s,y(\omega_{1}(s))\Big)d_{s}g_{1}(t_{1},s)\Big| \end{split}$$





$$+ |f_{1}(t_{1}, y(\omega_{1}(t_{1}))) \int_{0}^{t_{2}} u_{1}(t_{2}, s, y(\omega_{1}(s))) d_{s}g_{1}(t_{1}, s)|$$

$$- |f_{1}(t_{1}, y(\omega_{1}(t_{1}))) \int_{0}^{t_{2}} u_{1}(t_{1}, s, y(\omega_{1}(s))) d_{s}g_{1}(t_{1}, s)|$$

$$+ |f_{1}(t_{1}, y(\omega_{1}(t_{1}))) \int_{0}^{t_{2}} u_{1}(t_{1}, s, y(\omega_{1}(s))) d_{s}g_{1}(t_{1}, s)|$$

$$- |f_{1}(t_{1}, y(\omega_{1}(t_{1}))) \int_{0}^{t_{1}} u_{1}(t_{1}, s, y(\omega_{1}(s))) d_{s}g_{1}(t_{1}, s)|$$

$$\leq \aleph(h_{1}, \epsilon) + m_{1}(t_{2})|y(t_{2}) - y(t_{1})| \int_{0}^{t_{2}} |u_{1}(t_{2}, s, y(\omega_{1}(s)))| d_{s}(\bigvee_{p=0}^{s} g_{1}(t_{2}, p))$$

$$+ |f_{1}(t_{2}, y(t_{1})) - f_{1}(t_{1}, y(t_{1}))| \int_{0}^{t_{2}} |u_{1}(t_{2}, s, y(\omega_{1}(s)))| d_{s}(\bigvee_{p=0}^{s} g_{1}(t_{2}, p))$$

$$+ |f_{1}(t_{1}, y(t_{1}))| \int_{0}^{t_{2}} |u_{1}(t_{2}, s, y(\omega_{1}(s)))| d_{s}(\bigvee_{p=0}^{s} [g_{1}(t_{2}, p) - g_{1}(t_{1}, p)])$$

$$+ |f_{1}(t_{1}, y(t_{1}))| \int_{0}^{t_{2}} |u_{1}(t_{2}, s, y(\omega_{1}(s)))| d_{s}(\bigvee_{p=0}^{s} [g_{1}(t_{2}, p) - g_{1}(t_{1}, p)])$$

$$+ |f_{1}(t_{1}, y(t_{1}))| \int_{0}^{t_{2}} |u_{1}(t_{2}, s, y(\omega_{1}(s)))| d_{s}(\bigvee_{p=0}^{s} [g_{1}(t_{2}, p) - g_{1}(t_{1}, p)])$$

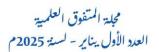
$$+ |f_{1}(t_{1}, y(t_{1}))| \int_{t_{1}}^{t_{2}} |u_{1}(t_{1}, s, y(\omega_{1}(s)))| d_{s}(\bigvee_{p=0}^{s} g_{1}(t_{1}, p))$$

$$\leq \aleph(h_{1}, \epsilon) + m_{1}(t_{2}) \aleph(y, \epsilon) \int_{0}^{t_{2}} n_{1}(t_{2}, s) \varphi_{1}(|y(s)|) d_{s}(\bigvee_{p=0}^{s} g_{1}(t_{2}, p))$$

$$+ \aleph_{f_{1}}(\epsilon) \int_{0}^{t_{2}} n_{1}(t_{2}, s) \varphi_{1}(|y(s)|) d_{s}(\bigvee_{p=0}^{s} g_{1}(t_{2}, p) - g_{1}(t_{1}, p)]$$

$$+ |m_{1}(t_{1})|y(t_{1})| + |f_{1}(t_{1}, 0)| \int_{0}^{t_{2}} n_{1}(t_{2}, s) \varphi_{1}(|y(s)|) d_{s}(\bigvee_{p=0}^{s} [g_{1}(t_{2}, p) - g_{1}(t_{1}, p)]$$

$$+ |m_{1}(t_{1})|y(t_{1})| + |f_{1}(t_{1}, 0)| \int_{0}^{t_{2}} \aleph_{u_{1}}(\epsilon) d_{s}(\bigvee_{p=0}^{s} g_{1}(t_{1}, p))$$





$$+[m_1(t_1)|y(t_1)| + \\ |f_1(t_1,0)|] \int_{t_1}^{t_2} n_1(t_1,s) \varphi_1(|y(s)|) d_s(\bigvee_{p=0}^s g_1(t_1,p))$$

Where

$$\aleph(h_i,\epsilon) = \sup\{|h_1(t_2) - h_1(t_1)|: t_1,t_2 \in I, |t_2 - t_1| < \epsilon, i = 1,2\},$$

$$\begin{split} \aleph_{fi}(\epsilon) &= \sup \left\{ |f_i(t_2,v) - f_i(t_1,v)| : \ t_1,t_2 \in I, |t_2 - t_1| < \\ \epsilon,v \in R, i = 1,2 \right\}, \\ \aleph_{ui}(\epsilon) &= \left\{ \left| u_1 \left( t_2,s,v(s) \right) - u_2(t_1,s,v(s)) \right| : \ t_1,t_2 \in I, |t_2 - t_1| < \epsilon,v \in R, i = 1,2 \right\}. \end{split}$$

Then, form estimate we get

$$|T_1y(t_2) - T_1y(t_1)| \le \aleph(h_1, \epsilon) + [m_1(t_2)\aleph(y, \epsilon)\aleph_{fi}(\epsilon)]n_1\varphi_1(||y||) + \int_0^{t_2} d_s(\bigvee_{p=0}^s g_1(t_2, p))$$

$$+ \left[ m_1 \, \|y\| \, + \right. \\ \left. F_1 \right] \left[ n_1 \varphi_1(\||\, y|\|) \int_0^{t_2} d_s \left( \bigvee_{p=0}^s [g_1(t_2,p) - g_1(t_1,p)] \right) \right. \\ \left. + \left[ m_1 \, \|y\| + \right] \left[ n_1 \varphi_1(\||\, y\|\|) \int_0^{t_2} d_s \left( \bigvee_{p=0}^s [g_1(t_2,p) - g_1(t_1,p)] \right) \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \left[ n_1 \varphi_1(\||\, y\|\|) \int_0^{t_2} d_s \left( \bigvee_{p=0}^s [g_1(t_2,p) - g_1(t_1,p)] \right) \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \left[ n_1 \varphi_1(\||\, y\|\|) \int_0^{t_2} d_s \left( \bigvee_{p=0}^s [g_1(t_2,p) - g_1(t_1,p)] \right) \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \left[ n_1 \varphi_1(\||\, y\|\|) \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \left[ n_1 \varphi_1(\||\, y\|\|) \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \left[ n_1 \varphi_1(\||\, y\|\|) \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \left[ n_1 \varphi_1(\||\, y\|\|) \right] \\ \left[ m_1 \, \|y\| + \right] \left[ m_1 \, \|y\| + \right] \right] \\ \left[ m_1 \, \|y\| + \right] \\ \left[ m_1$$

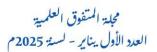
$$\begin{split} + \aleph_{u1}(\epsilon) \int_0^{t_2} d_s \Big( \bigvee_{p=0}^s g_1(t_2, p) \Big) + n_1 \varphi_1(\||y|\|) \int_{t_1}^{t_2} d_s \Big( \bigvee_{p=0}^s g_1(t_1, p) \Big) \Big] \\ & \leq \aleph(h_1, \epsilon) + [m_1(t_2) \aleph(y, \epsilon) \aleph_{fi}(\epsilon)] n_1 \varphi_1(\||y|\|) + 0 \end{split}$$

$$\int_0^{t_2} d_s(\bigvee_{s=0}^t g_1(t_2, s))$$

$$+ \left[ m_1 \, \|y\| + \right. \\ \left. F_1 \right] \left[ n_1 \varphi_1(\||y|\|) \int_0^{t_2} d_s(\bigvee_{s=0}^t [g_1(t_2,s) - g_1(t_1,s)]) \right. \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right] \right] \\ \left. + \left[ m_1 \, \|y\| + \right]$$

$$+ \aleph_{u1}(\epsilon) \int_0^{t_2} d_s(\bigvee_{s=0}^s g_1(t_2, s)) + n_1 \varphi_1(|||y|||) \int_{t_1}^{t_2} d_s(\bigvee_{s=0}^t g_1(t_1, s)) \Big]$$

$$\leq \aleph(h_1, \epsilon) + K_1[m_1 \aleph(y, \epsilon) + \aleph_{f1}(\epsilon)] n_1 \varphi_1(r)$$





$$+ [m_1 r + F_1][n_1 \varphi_1(r) W_1(\epsilon) + 
onumber \ \, N_{f1}(\epsilon)[g_1(t_1, t_2) - g_1(t_1, 0)] 
onumber \ \, + n_1 \varphi_1(r)[g_1(t_1, t_2) - g_1(t_1, t_1)]].$$

Hence, from the continuity of the functions  $g_1$  assumption (vi), we deduce that  $T_1$  maps C(I) into C(I). As done above we can obtain

$$\begin{split} |T_2y(t_2)-T_2y(t_1)| &\leq \aleph(h_2,\epsilon) + K_2[m_2\aleph(y,\epsilon) + \aleph_{f2}(\epsilon)]n_2\varphi_2(r) \\ &\qquad + [m_2\,r + F_2] \big[n_2\varphi_2(r)W_2(\epsilon) + \\ \aleph_{f2}(\epsilon)[g_2(t_1,t_2) - g_2(t_1,0)] \\ &\qquad + n_2\varphi_2(r)[g_2(t_1,t_2) - g_2(t_1,t_1)] \big] \end{split}$$

Also, by our assumption (iv), we see that  $T_2$  maps C(I) into C(I). Now, from the definition of the operator T we get

$$\begin{split} Tv(t_2) - Tv(t_1) &= T(x, y)(t_2) - T(x, y)(t_1) \\ &= \left(T_1 y(t_2), T_2 x(t_2)\right) - \left(T_1 y(t_1), T_2 x(t_1)\right) \\ &= \left(T_1 y(t_2) - T_1 y(t_1), T_2 x(t_2) - T_2 x(t_1)\right). \end{split}$$

Therefore

$$\begin{split} \|Tv(t_2) - Tv(t_1)\| &= \left\| \left( T_1 y(t_2) - T_1 y(t_1), T_2 x(t_2) - T_2 x(t_1) \right) \right\| \\ &= \max \left\{ \|T_1 y(t_2) - T_1 y(t_1)\|, \ \|T_2 y(t_2) - T_2 y(t_1)\| \right\} \\ &\leq \max \left\{ \aleph(h_1, \epsilon) + K_1 [m_1 \aleph(y, \epsilon) + \Re_{f1}(\epsilon)] n_1 \varphi_1(r) \right\} \\ &+ [m_1 r + F_1] [n_1 \varphi_1(r) W_1(\epsilon) + \Re_{u1}(\epsilon) [g_1(t_1, t_2) - g_1(t_1, 0)] \end{split}$$



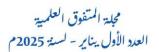
$$egin{align*} &+n_1 arphi_1(r) [g_1(t_1,t_2)-g_1(t_1,t_1)]], \ && \& (h_2,\epsilon) + K_2 [m_2 \& (y,\epsilon) + \&_{f2}(\epsilon)] n_2 arphi_2(r) \ && + [m_2 r + F_2] [n_2 arphi_2(r) W_2(\epsilon) + \&_{u2}(\epsilon) [g_2(t_1,t_2) - g_2(t_1,0)] \ && + n_2 arphi_2(r) [g_2(t_1,t_2) - g_2(t_1,t_1)]] \}. \end{split}$$

This means that the class of  $\{Tv(t)\}$  is equi-continuous on I, then by Arzela-Ascoil theorem  $\{Tv(t)\}$  is relatively compact.

Now, we will show that the operator  $T: V \rightarrow V$  is continuous.

Firstly, we prove that  $T_1$  is continuous. Let  $\epsilon * > 0$ , the continuity of  $u_i$ , i = 1, 2, yields,  $\exists \delta = \delta(\epsilon *)$  such that  $|u_i(t, s, u(s)) - u_i(t, s, v(s))| < \epsilon *$ , whenever  $||u - v|| \le \delta$ , thus if  $||y - u|| \le \delta$ , we arrive at:

$$\begin{split} &|(T_{1}y)(t) - (T_{1}u)(t)| \\ &\leq \Big| f_{1}(t,y(\omega_{1}(t)) \int_{0}^{t} u_{1}(t,s,y(\omega_{1}(s)) d_{s}g_{1}(t,s) \\ &- f_{1}(t,u(\omega_{1}(t)) \int_{0}^{t} u_{1}(t,s,u(\omega_{1}(s)) d_{s}g_{1}(t,s) \Big| \\ &\leq \Big| f_{1}(t,y(\omega_{1}(t)) \int_{0}^{t} u_{1}(t,s,u(\omega_{1}(s)) d_{s}g_{1}(t,s) \\ &- f_{1}(t,u(\omega_{1}(t)) \int_{0}^{t} u_{1}(t,s,y(\omega_{1}(s)) d_{s}g_{1}(t,s) \Big| \\ &+ f_{1}(t,u(\omega_{1}(t)) \int_{0}^{t} u_{1}(t,s,y(\omega_{1}(s)) d_{s}g_{1}(t,s) \\ &- f_{1}(t,u(\omega_{1}(t)) \int_{0}^{t} u_{1}(t,s,u(\omega_{1}(s)) d_{s}g_{1}(t,s) \Big| \\ &\leq \Big| f_{1}(t,y(\omega_{1}(t)) - f_{1}(t,u(\omega_{1}(t)) \Big| \int_{0}^{t} \Big| u_{1}(t,s,y(\omega_{1}(s)) - u_{1}(t,s,u(\omega_{1}(s)) \Big| d_{s}g_{1}(t,s) \\ &+ |f_{1}(t,u(\omega_{1}(t)) | \int_{0}^{t} \Big| u_{1}(t,s,y(\omega_{1}(s)) - u_{1}(t,s,u(\omega_{1}(s)) \Big| d_{s}g_{1}(t,s) \end{split}$$





$$\leq m_{1}(t)|y(t) - u(t)| \int_{0}^{t} n_{1}(t,s)\varphi_{1}(|y(s)|) d_{s}g_{1}(t,s) + [m_{1}(t)|u(t_{1})| + |f_{1}(t_{1},0)|] \int_{0}^{t} |u_{1}(t,s,y(\omega_{1}(s)) - u_{1}(t,s,u(\omega_{1}(s)))| d_{s}g_{1}(t,s)$$

$$\leq (\delta m_{1}n_{1}\varphi_{1}(||y||) + [m_{1}||u|| + F_{1}]\epsilon^{*}) \int_{0}^{t} d_{s} \bigvee_{p=0}^{s} g_{1}(t,p)$$

$$\leq (\delta m_{1}n_{1}\varphi_{1}(||y||) + [m_{1}||u|| + F_{1}]\epsilon^{*}) \bigvee_{s=0}^{t} g_{1}(t,s),$$

$$\leq (\delta m_{1}n_{1}\varphi_{1}(||y||) + [m_{1}||u|| + F_{1}]\epsilon^{*}) K_{1},$$

where

$$\epsilon = (\delta m_1 n_1 \varphi_1(||y||) + [m_1 ||u|| + F_1] \epsilon^*) K_1.$$

Therefore

$$|(T_1y)(t)-(T_1u)(t)|\leq \epsilon.$$

This means that the operator  $T_1$  is continuous.

By a similar way as done above we can prove that for any  $x, u \in C[0,T]$  and  $\|x - v\| < \delta$ , we have

$$|(T_2x)(t)-(T_2v)(t)|\leq \epsilon.$$

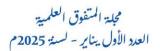
Hence  $T_2$  is continuous operator. The operators  $T_i$  (i = 1, 2) are continuous operator this imply that T is continuous operator. Since all conditions of Schauder fixed point theorem are satisfied, then T has at least one fixed point  $v = (x, y) \in V$ , which completes the proof

**Corollary 1.** Let assumptions of Theorem 1 be satisfied. Then quadratic Volterra-Stieltjes functional integral equation

$$x(t) = h(t) + f(t, y(\omega(t))) \int_0^t u(t, s, y(\omega(s))) d_s g(t, s),$$

$$t \in I$$
 (3)

has at least one solution  $x \in C(I)$ .





**Proof.** Let the assumptions of Theorem 1 be satisfied. With x = y,  $f_1 = f_2 = f$ ,  $u_1 = u_2 = u$ , and  $h_1 = h_2 = h$ . Then the coupled system (1) will be the Volterra-Stieltjes quadratic integral equation (3)

### 3. Existence of unique solution

Here, we study the uniqueness of the solution  $(x, y) \in X$  of the coupled system of

quadratic Volterra-Stieltjes integral equations (1). Assume that functions  $\varphi_i$ :  $R+\to R$  + have the

form  $\varphi_i(x) = 1 + |x|$ , and the functions  $n_i(t,s) \in C(I)$  denoted by  $b_i = \|n_i\| = \max\{n_i(t,s) \ t,s \in I, i = 1,2\}$ . Then

$$|u_i(t,s,x)| \leq n_i(t,s)(1+|x|).$$

Notice that this assumption is a special case of assumption (iii). Consider now the assumptions (ii)\*, (iii)\* having the form

(ii)\*  $f_i$ :  $I \times R \to R$  are continuous functions and there exist constants numbers  $m_i$  such that

$$|f_i(t,x)-f_i(t,y)| \le m_i|x-y|, \quad i=1,2.$$

From this assumption, we can deduce that

$$|u_i(t,s,x)| - |u_i(t,s,0)| \le |u_i(t,s,x) - u_i(t,s,0)| \le b_i|x|$$

which implies that

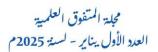
$$|u_i(t,s,x)| \le |u_i(t,s,0)| + b_i|x| \le n_i(t,s) + b_i|x|,$$

where 
$$n_i(t,s) = \sup_{t \in I} |u_i(t,s,0)|$$
.

**Theorem 2.** Let assumptions of Theorem 1 be satisfied with replace assumptions (ii), (iii) by (ii)\*, (iii)\*, if the following conditions hold

$$m(n+br)+(mr+F)K\leq 1.$$

Then the coupled system (1) has an unique solution  $(x, y) \in X$ .





**Proof.** Let  $v_1 = (x_1, y_1)$  and  $v_2 = (x_2, y_2)$  be two solutions of the coupled system (1), we have

$$\|(x_1, y_1) - (x_2, y_2)\|_X = \|(x_1 - x_2, y_1 - y_2)\|_X$$
$$= \max_{t \in I} \{\|x_1 - x_2\|, \|y_1 - y_2\|\}.$$

Now

$$|x_{1}(t) - x_{2}(t)|$$

$$\leq |f_{1}(t, y_{1}(\omega_{1}(t)) \int_{0}^{t} u_{1}(t, s, y_{1}(\omega_{1}(s)) d_{s}g_{1}(t, s))|$$

$$- f_{1}(t, y_{2}(\omega_{1}(t)) \int_{0}^{t} u_{1}(t, s, y_{2}(\omega_{1}(s)) d_{s}g_{1}(t, s))|$$

$$\leq |f_{1}(t, y_{1}(\omega_{1}(t)) \int_{0}^{t} u_{1}(t, s, y_{1}(\omega_{1}(s)) d_{s}g_{1}(t, s))|$$

$$- f_{1}(t, y_{2}(\omega_{1}(t)) \int_{0}^{t} u_{1}(t, s, y_{1}(\omega_{1}(s)) d_{s}g_{1}(t, s))|$$

$$+ |f_{1}(t, y_{2}(\omega_{1}(t)) \int_{0}^{t} u_{1}(t, s, y_{1}(\omega_{1}(s)) d_{s}g_{1}(t, s))|$$

$$+ |f_{1}(t, y_{2}(\omega_{1}(t)) \int_{0}^{t} u_{1}(t, s, y_{2}(\omega_{1}(s)) d_{s}g_{1}(t, s))|$$

$$\leq |f_{1}(t, y_{1}(\omega_{1}(t)) - f_{1}(t, y_{2}(\omega_{1}(t))) \int_{0}^{t} |u_{1}(t, s, y_{1}(\omega_{1}(s)) - u_{1}(t, s, y_{2}(\omega_{1}(s))) | d_{s}g_{1}(t, s))|$$

$$+ |f_{1}(t, y_{2}(\omega_{1}(t))) \int_{0}^{t} |u_{1}(t, s, y_{1}(\omega_{1}(s)) - u_{1}(t, s, y_{2}(\omega_{1}(s))) | d_{s}g_{1}(t, s)|$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t))) | d_{s}g_{1}(t, s)|$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) | |f_{0}^{t}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) | |f_{0}^{t}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) | |f_{0}^{t}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) | |f_{0}^{t}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) | |f_{0}^{t}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) | |f_{0}^{t}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) | |f_{0}^{t}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) |f_{0}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

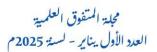
$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) |f_{0}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) |f_{0}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) |f_{0}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) |f_{0}(u_{1}(t, s) + b_{1}|y_{1}|) d_{s}g_{1}(t, s)$$

$$\leq |f_{1}(t, y_{2}(\omega_{1}(t)) |f_{0}(u_{1}(t, s) + b_{1}|y_{1}|y_{1}| d_{s}(u_{1}(t, s) + b_{1}|y_{1}|y_{1}| d_{s}(u_{1}(t, s) + b_{1}|y_{1}|y_{1}| d_{s}(u$$





$$\leq m_1(n_1+b_1r_1)+[m_1r_1+F_1]K_1||y_1-y_2||.$$

Therefore

$$||x_1 - x_2|| \le m(n + br) + [mr + F]K||y_1 - y_2||,$$

where

$$b = max\{b_1, b_2\}, \ m = max\{m_1, m_2\}, \ n = max\{n_1, n_2\}, \ F = max\{F_1, F_2\} \text{ and } K = max\{K_1, K_2\}.$$

Similarly

$$||y_1 - y_2|| \le m(n + br) + [mr + F]K||x_1 - x_2||.$$

Then

$$\|(x_1, y_1) - (x_2, y_2)\|_X = \|(x_1 - x_2, y_1 - y_2)\|_X$$
$$= \max_{t \in I} \{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}$$

$$= \max_{t \in I} \{ m(n+br) + [mr+F]K \| y_1 - y_2 \| m(n+br) + [mr+F]K \| x_1 - x_2 \| \}$$

$$= [m(n+br) + [mr+F]K] \max_{t \in I} \{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}$$
$$= m(n+br) + [mr+F]K \|(x_1, y_1) - (x_2, y_2)\|_X.$$

Which implies that

$$[1-m(n+br)+[mr+F]K]\|(x_1,y_1)-(x_2,y_2)\|_X\leq 0.$$

This means that

$$(x_1, y_1) = (x_2, y_2) \implies x_1 = x_2, \qquad y_1 = y_2.$$

This proves the uniqueness of the solution of the coupled system (1).

#### 4. SPECIAL CASES

In this section, we will consider a coupled system of quadratic Volterraintegral equations of fractional order, which has form

$$x(t) = h_1(t) + f_1(t, y(\omega_1(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha_1)} u_1(t, s, y(\omega_1(t))) ds, \ t \in I$$



**(4)** 

$$y(t) = h_1(t) + f_2(t, x(\omega_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha_1)} u_2(t, s, x(\omega_2(t))) ds, \ t \in I$$

where  $t \in I = [0,T]$  and  $\alpha_i \in (0,1)$ , and  $\Gamma(\alpha_i)$ , i = 1,2, refers to gamma functions. Let us mention that (4) represents the so-called a coupled systems of Volterra quadratic integral equations of fractional order. Recently, such a type this type has been widely investigated in some papers [1, 9, 10, 12, 13, 14] Here, we show that a coupled systems of fractional orders (4) can be treated as a special case of a coupled systems of quadratic Volterra-Stieltjes integral equations (2) studied in Section 2.

In fact, we can consider functions  $g_i(t,s) = g_i : \triangle \to R$ , i = 1,2, defined by the formula

$$g_i(t,s) = \frac{t^{\alpha_i} - (t-s)^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

We can see that functions  $g_i$ , i = 1, 2, satisfy assumptions (vi)-(vii) in Theorem 1, see [6, 8].

Now, we state the following existence results for couple system of quadratic Volterra integral equations of fractional order (4).

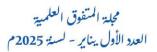
**Theorem 3.** Let assumptions (i)-(viii) of Theorem 1 be satisfied. Then a coupled systems of fractional orders (4) has at least one solution  $(x, y) \in X$ .

Corollary 2. Let assumptions of Theorem 3 be satisfied (with  $= y, u_1 = u_2 = u$ ,  $f_1 = f_2 = f, h_1 = h_2 = h$  and  $\alpha_1 = \alpha_2 = \alpha$ ). Then the fractional-order quadratic integral equation

$$x(t) = h(t) + f(t,x(\omega(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(t,s,x(\omega(t))) ds, \ t \in I$$

has at least one solution in  $x \in C(I)$ .

Corollary 3. Let assumptions of Theorem 3 be satisfied, with  $f_1(t, y(t)) = f_2(t, x(t)) = 1$ . Then a coupled system of the fractional-order quadratic integral system





$$x(t) = h_1(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_1(t,s,y(\omega_1(t))) ds, \qquad t \in I$$

$$y(t) = h_1(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} u_2(t,s,x(\omega_2(t))) ds, \qquad t \in I$$

has at least one solution in  $(x, y) \in X$ .

Now, letting  $\alpha_1, \alpha_2 \rightarrow 1$ , we obtain

**Corollary 4.** Let assumptions of Theorem 3 be satisfied. Then the coupled system of the initial value problems

$$\frac{x(t)}{dt} = u_1 \left( t, s, y(\omega_1(t)) \right), \qquad t \in I, \quad x(0) = x_0,$$

$$\frac{y(t)}{dt} = u_1 \left( t, s, x(\omega_2(t)) \right), \qquad t \in I, \quad y(0) = y_0,$$
(6)

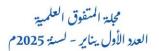
**Proof.** Let assumptions of Theorem 3 be satisfied (with  $f_1(t, y(t)) = f_2(t, x(t)) = 1$ ,  $h_1(t) = x_0$ ,  $h_2(t) = y_0$  and letting  $\alpha, \beta \to 1$ . Then a coupled system of the fractional-order quadratic integral equations

$$x(t) = x_0 + \int_0^t u_1(t, s, y(\omega_1(t))) ds, \qquad t \in I,$$

$$y(t) = y_0 + \int_0^t u_1(t, s, x(\omega_2(t))) ds, \qquad t \in I,$$
(7)

has at least one solution in X which is equivalent to the coupled system of the initial value problems (6).

**Corollary 5.** Let assumptions of Theorem 3 be satisfied. Then the coupled system of fractional order differential equations





$$D^{\alpha_1}x(t) = u_1(t, s, y(\omega_1(t))), \qquad t \in I$$

$$D^{\alpha_2}y(t) = u_2(t, s, x(\omega_2(t))), \qquad t \in I,$$
(8)

with the initial conditions

$$I^{1-\alpha_1}x(t)\,\big|_{\,t=0}=I^{1-\alpha_2}y(t)\,\big|_{\,t=0}=0,\,\,\alpha_1,\alpha_2\in(0,1],$$

has at least one solution in  $(x, y) \in X$ .

**Proof.** let us proof the coupled system of the initial value problems (8) and (9) is equivalent to the coupled system of quadratic integral system

$$x(t) = \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_{1}(t,s,y(\omega_{1}(t))) ds, \qquad t \in I$$

$$y(t) = \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} u_{2}(t,s,x(\omega_{2}(t))) ds, \qquad t \in I$$

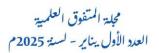
$$(10)$$

By operating  $I^{1-\alpha_1}$  and  $I^{1-\alpha_2}$  respectively on each equation of coupled system (10), and applying properties of fractional operator [23], we obtain

$$I^{1-\alpha_1}x(t) = I^1u_1(t,s,y(\omega_1(t))), \qquad I^{1-\alpha_1}x(t) \Big|_{t=0} = 0$$
 
$$I^{1-\alpha_2}y(t) = I^1u_2(t,s,y(\omega_2(t))), \qquad I^{1-\alpha_2}y(t) \Big|_{t=0} = 0.$$

Also,

$$\frac{d}{dt}I^{1-\alpha_1}x(t)=u_1\Big(t,s,y\Big(\omega_1(t)\Big)\Big),\ t\in I,\ \alpha_1\in(0,1)$$





$$\frac{d}{dt}I^{1-\alpha_2}y(t)=u_2\Big(t,s,y\Big(\omega_2(t)\Big)\Big),\ t\in I,\ \alpha_2\in(0,1).$$

Conversely, by integrating the coupled system of initial value problems (8) and (9), we have

$$I^{1-\alpha_1}x(t) - I^{1-\alpha_1}x(t) \Big|_{t=0} = I^1u_1(t, s, y(\omega_1(t)))$$

$$I^{1-\alpha_2}y(t) - I^{1-\alpha_2}y(t) \Big|_{t=0} = I^1u_2(t, s, y(\omega_2(t))).$$

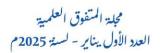
Operating by  $I^{\alpha_1}$  and  $I^{\alpha_2}$  respectively on each equation and differentiating, we have (10). Thus, the equivalence hold.

Let assumptions of Theorem 3 be satisfied (with  $f_1(t, y(\omega_1(t))) =$ 

 $f_2(t, x(\omega_2(t))) = 1, h_1(t) = h_2(t) = 0$ . Then there exists at least one solution in X for the coupled system (8 and 9).

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