Analysis Of A Coupled Quadratic Volterra-Stieltjes Integral Equation System

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Abstract:

The purpose of this study is to prove actuality findings for a coupled system of Volterra- Stieltjes type nonlinear quadratic integral equations over a unrestricted bounded interval in the space of nonstop functions.

key words: function of bounded variation, fractional order, coupled system, and nonlinear quadratic integral equation

1. Introduction

In the applied lores, integral equations are essential for bluffing a variety of events and marvels. Functional analysis, topology, and fixed point proposition have all been used to make substantial advancements in this discipline (see (1, 9, 10, 11, 12, 15)).

Examining the solvability of a coupled system of quadratic Volterra- Stieltjes integral equations is the main thing of this work.

We calculate on the delineations and parcels presented by Banas (see(2, 3)) to explain the significance of the Stieltjes integral in this environment.

also, previous exploration on analogous areas has inspired interest in probing similar coupled systems.

The actuality theorems for a linked system of quadratic Volterra- Stieltjes integral equations, which include several types of Volterra integral equations as special cases, are established in this paper. We decide actuality results under general and flexible hypotheticals thanks to our evidence, which is grounded on the fixed- point conception.

Throughout this paper, let $\boldsymbol{I} = (\mathbf{0}, \boldsymbol{T})$ and $\boldsymbol{X}$ be the Banach space of all ordered dyads $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} = \boldsymbol{C}(\boldsymbol{I}) \times \boldsymbol{C}(\boldsymbol{I})$, with the norm

$$\|(x, y)\|_X = max\{\|x\|_C, \|y\|_C\},$$

where

$$\|x\|_C = \sup_{t\in I}|x|, \|y\|_C = \sup_{t\in I}|y|,$$

It's clear that $(X, \|(x, y)\|, X)$ is Banach space.

Now, we shall present some auxiliary properties of fractional calculus that will be need in this work.

Definition 1. The Riemann-Liouville of a fractional integral of the function $f \in L\_1(I)$ of order $\alpha \in R^+$ is defined by

$$I_a^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

and when $h = 0$, we have $I^\alpha f(t) = I_a^\alpha f(t).$

Definition 2. The (Caputo) fractional order derivative $D^\alpha, \alpha \in (0, 1]$ of the absolutely continuous function g is defined as

$$D_a^\alpha g(t) = I_a^{1-\alpha} \frac{d}{dt} g(t), \qquad t \in [a, b].$$

For farther parcels of fractional math driver(See (20),(21),(22) and (23)).

2. Preliminaries

In this section, we study the solvability of the coupled system of nonlinear quadratic integral equations of Volterra- Stieltjes type having the form

$$x(t) = h_1(t) + f_1(t, y(\omega_1(t))) \int_0^t u_1(t, s, y(\omega_1(s)) d_s g_1(t, s), \qquad t \in I$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad (1)$$

$$y(t) = h_2(t) + f_{12}(t, y(\omega_2(t))) \int_0^t u_2(t, s, y(\omega_2(s)) d_s g_2(t, s), \qquad t \in I$$

Our thing is to show that system (1) has at least one result in the space $\boldsymbol{X}$. For our farther purposes we denote by $\triangle$ the triangle $\triangle = \{(\boldsymbol{t},\boldsymbol{s})\ \mathbf{0} \leq \boldsymbol{s} \leq \boldsymbol{t} \leq \boldsymbol{T}\}$.

In our investigations, , we give some hypotheticals which are demanded throughout this paper.

(i) $\boldsymbol{h_i}: \boldsymbol{I} \to \boldsymbol{R}, (\boldsymbol{i} = \mathbf{1},\mathbf{2})$ are continuous on $\boldsymbol{I}$. There are constants $\boldsymbol{h_i}$, where $\boldsymbol{h_i} = \sup_{t\in I}|\boldsymbol{h_i}(\boldsymbol{t})|$.

(ii) $\boldsymbol{fi} : \boldsymbol{I} \times \boldsymbol{R} \to \boldsymbol{R}, (\boldsymbol{i} = \mathbf{1},\mathbf{2})$ are continuous functions and there exist continuous functions

$\boldsymbol{m_i}(\boldsymbol{t}) : \boldsymbol{I} \to \boldsymbol{I}$ such that

$$|\boldsymbol{f_i}(\boldsymbol{t},\boldsymbol{x}) - \boldsymbol{f_i}(\boldsymbol{t},\boldsymbol{y})| \leq \boldsymbol{m_i}(\boldsymbol{t})|\boldsymbol{x} - \boldsymbol{y}|,$$

for all $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{R}$ and $\boldsymbol{t} \in \boldsymbol{I}$. Moreover, we put $\boldsymbol{m_i} = \boldsymbol{max}\{\boldsymbol{m_i}(\boldsymbol{t})\ \boldsymbol{t} \in \boldsymbol{I},\}$.

(iii) $\boldsymbol{u_i}(\boldsymbol{t},\boldsymbol{s},\boldsymbol{x}) : \triangle \times \boldsymbol{R} \to \boldsymbol{R}, (\boldsymbol{i} = \mathbf{1},\mathbf{2})$ are continuous functions and there exist continuous functions $\boldsymbol{n_i}(\boldsymbol{t},\boldsymbol{s}) : \triangle \to \boldsymbol{I}$, and continuous and nondecreasing functions $\boldsymbol{\varphi_i} : \boldsymbol{R}+ \to \boldsymbol{R}+$, such that

$$|\boldsymbol{u_i}(\boldsymbol{t},\boldsymbol{s},\boldsymbol{x})| \leq \boldsymbol{n_i}(\boldsymbol{t},\boldsymbol{s})\boldsymbol{\varphi_i}(|\boldsymbol{x}|),$$

for all $(\boldsymbol{t},\boldsymbol{s}) \in \triangle$ and $\boldsymbol{x} \in \boldsymbol{R}$. Moreover, we put $\boldsymbol{n_i} = \boldsymbol{max}\{\boldsymbol{n_i}(\boldsymbol{t},\boldsymbol{s})\ \boldsymbol{t},\boldsymbol{s} \in \boldsymbol{I}\}$.

(iv) $\omega_i : I \to I$ are continuous, $(\boldsymbol{i} = \mathbf{1},\mathbf{2})$.

(v) Functions $\boldsymbol{s} \to \boldsymbol{g_i}(\boldsymbol{t},\boldsymbol{s})$ are of bounded variation on $[\mathbf{0},\boldsymbol{t}]$ for each $\boldsymbol{t} \in \boldsymbol{I}, \boldsymbol{i} = \mathbf{1},\mathbf{2}$.

(vi) Functions $\boldsymbol{g_i}$ are continuous on the triangle $\triangle$ and $\boldsymbol{g_i}(\boldsymbol{t},\mathbf{0}) = \mathbf{0}$ for $\boldsymbol{i} = \mathbf{1},\mathbf{2}$.

(vii) $\boldsymbol{g_i}(\boldsymbol{t},\boldsymbol{s}) = \boldsymbol{g_i} : \Delta_i \to \boldsymbol{R}, \boldsymbol{i} = \mathbf{1},\mathbf{2}$ and for all $\boldsymbol{t_1},\boldsymbol{t_2} \in \boldsymbol{I}$ with $\boldsymbol{t_1} < \boldsymbol{t_2}$, the functions $\boldsymbol{s} \to \boldsymbol{g_i}(\boldsymbol{t_2},\boldsymbol{s}) - \boldsymbol{g_i}(\boldsymbol{t_1},\boldsymbol{s})$ are nondecreasing on I.

(viii) For any $\epsilon > 0$ there exists $\delta > 0$ such that, for all$t_1; t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$

the following inequality holds

$$\bigvee_{s=0}^{t}[g_i(t_2,s) - g_i(t_1,s)] \leq \epsilon, \quad i = 1,2.$$

Obviously, we will assume that $g_i, (i = 1,2)$ satisfy hypotheticals $(iv) - (vii)$. For our purposes, we will need the following lemmas.

Lemma 1. [6] Under hypotheticals (v)-(viii), The functions $z \to \bigvee_{s=0}^{z} g_i(t,s)$ are continuous on $[0,t]$ for any $t \in I$ $(i = 1,2)$.

Lemma 2. [6] Let hypotheticals (v)-(viii) be satisfied. Then, for arbitrary fixed number $0 < t_2 \in I$

and for any $\epsilon > 0$, there exists $\delta > 0$ such that if $t_1 \in I$; $t_1 < t_2$ and$t_2 - t_1 \leq \delta$ then $\bigvee_{s=t_1}^{t_2} g_i(t_2,s) \leq \epsilon$. $(i = 1,2)$.

Further, let us observe that grounded on Lemma 1 we infer that there exists finite positive constants

K_i, similar that

$$K_i = sup\left\{\bigvee_{s=0}^{t} g_i(t,s): t \in [0,T]\right\}.$$

where $T > 0$ is arbitrarily fixed $(i = 1,2)$.

We now introduce some functions that will be useful in our further studies:

$$W_i = sup\left\{\bigvee_{s=0}^{t_2}\big(g_i(t_2,s) - g_i(t_1,s)\big): \ t_1,\ t_1 \in I, t_1 < t_2 \leq \epsilon, i = 1,2\right\}.$$

In what follows let us denote by F_i the constant defined by the formula:

$$F_i = sup\{|f_i(t,0)| : t \in I, i = 1,2\}.$$

Now, we are in position to present the main result of the paper.

3. Main Result

Defined the operator by

$$T(x,y)(t) = \left(T_1 y(t), T_2 x(t)\right),$$

where

$$T_1 x(t) = h_1(t) + f_1(t, y(\omega_1(t))) \int_0^t u_1(s, y(\omega_1(s))) d_s g_1(t,s), \qquad t \in I$$

$$(2)$$

$$T_2 y(t) = h_2(t) + f_{12}(t, y(\omega_2(t))) \int_0^t u_2(s, y(\omega_2(s))) d_s g_2(t,s), \qquad t \in I$$

Theorem 1. Let hypotheticals (i)-(vii) be satisfied. also the coupled system of quadratic Volterra- Stieltjes integral equations(1) has at least one solution (x,y) belonging to the space X.

Proof. We prove a many results concerning the continuity and compactness of these operators in the space of continuous functions.

Define

$$V = \{v = (x(t), y(t)) : (x(t), y(t)) \in X : \|(x,y)\|_X \le r\}.$$

For $(x,y) \in V$, and define the operator T map V into V, we have

$$|T_1 y(t)| \le |h_1| + |f_1(t, y(\omega_1(t)))| \int_0^t |u_1(t,s,y(\omega_1(s)))| |d_s g_1(t,s)|$$

$$\le \|h_1\| + [m_i|y(t)| + |f_1(t,0)|] \int_0^t n_1(t,s)\varphi_1(|y(s)|) d_s (\vee_{P=0}^t g_1(t,P),$$

$$\le \|h_1\| + [\|y\|m_1 + F_1] n_1 \varphi_1(\|y\|)(\vee_{p=0}^t g_1(t,p),$$

$$\|T_1 y\| \le \|h_1\| + [r_1 m_1 + F_1] n_1 \varphi_1(r_1) \sup_{t\in I}(\vee_{p=0}^t g_1(t,p)).$$

Hence, we get

$$\|T_1y\| \le \|h_1\| + K_1[m_1r_1 + F_1]n_1\varphi_1(r_1).$$

From the last estimate we conclude that $r_1 = \frac{\|h_1\| + F_1K_1n_1\varphi_1(r_1)}{1 - m_1n_1K_1\varphi_1(r_1)}$.

By a analogous way, we gain

$\|T_2y\| \le \|h_2\| + K_2[m_2r_2 + F_2]n_2\varphi_2(r_2).r_2 = \frac{\|h_2\| + F_2K_2n_2\varphi_2(r_2)}{1 - m_2n_2K_2\varphi_2(r_2)}$.

Thus

$$\|Tv\|_X = \|T(x,y)\|_X = \|T_1y, T_2x\|_X \le \max_{t\in I}\{\|T_1y\|_C, \|T_2y\|_C\} = r.$$

Therefore for every $v = (x,y) \in V$, we have $Tv \in V$ and hence $TV \subset V$, ($i.e\ T: V \to V$). This means that the functions of TU are uniformly bounded on I, it is clear that the set V is nonempty, bounded, closed and convex. Now, we need to show that the set TV is relatively compact.

For $v = (x,y) \in V$, for all $\epsilon > 0, \delta > 0$, and for each $t_1, t_2 \in I$(without loss of generality assume that $t_1 < t_2$, such that $|t_2 - t_1| < \delta$, we have

$$|T_1y(t_2) - T_1y(t_1)|$$

$$= \Big|h_1(t_2) - h_1(t_1) + f_1(t_2, y(\omega_1(t_2))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)$$

$$- f_1(t_1, y(\omega_1(t_1))) \int_0^{t_1} u_1\left(t_1, s, y(\omega_1(s))\right) d_sg_1(t_1, s)\Big|$$

$$\le |h_1(t_2) - h_1(t_1)| + |f_1(t_2, y(\omega_1(t_2))) - f_1(t_2, y(\omega_1(t_1)))|$$

$$. \left|\int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\right|$$

$$+ \Big|f_1(t_2, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\Big|$$

$$- \Big|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\Big|$$

$$+ \Big|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_2, s)\Big|$$

$$- \Big|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_sg_1(t_1, s)\Big|$$

$$+ \left|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_2, s, y(\omega_1(s))\right) d_s g_1(t_1, s)\right|$$

$$- \left|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_1, s, y(\omega_1(s))\right) d_s g_1(t_1, s)\right|$$

$$+ \left|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_2} u_1\left(t_1, s, y(\omega_1(s))\right) d_s g_1(t_1, s)\right|$$

$$- \left|f_1(t_1, y(\omega_1(t_1))) \int_0^{t_1} u_1\left(t_1, s, y(\omega_1(s))\right) d_s g_1(t_1, s)\right|$$

$$\leq \aleph(h_1, \epsilon) + m_1(t_2)|y(t_2) - y(t_1)| \int_0^{t_2} \left|u_1(t_2, s, y(\omega_1(s)))\right| d_s\left(\vee_{p=0}^{s} g_1(t_2, p)\right)$$

$$+ \left|f_1\left(t_2, y(t_1)\right) - f_1\left(t_1, y(t_1)\right)\right| \int_0^{t_2} \left|u_1\left(t_2, s, y(\omega_1(s))\right)\right| d_s\left(\vee_{p=0}^{s} g_1(t_2, p)\right)$$

$$+ \left|f_1\left(t_1, y(t_1)\right)\right| \int_0^{t_2} \left|u_1\left(t_2, s, y(\omega_1(s))\right)\right| d_s\left(\vee_{p=0}^{s}[g_1(t_2, p) - g_1(t_1, p)]\right)$$

$$+ \left|f_1\left(t_1, y(t_1)\right)\right| \int_0^{t_2} \left|u_1\left(t_2, s, y(\omega_1(s))\right) - u_1\left(t_1, s, y(\omega_1(s))\right)\right| d_s\left(\vee_{p=0}^{s} g_1(t_1, p)\right)$$

$$+ \left|f_1\left(t_1, y(t_1)\right)\right| \int_{t_1}^{t_2} \left|u_1(t_1, s, y(\omega_1(s)))\right| d_s\left(\vee_{p=0}^{s} g_1(t_1, p)\right)$$

$$\leq \aleph(h_1, \epsilon) + m_1(t_2)\aleph(y, \epsilon) \int_0^{t_2} n_1(t_2, s)\varphi_1(|\, y(s)|) d_s\left(\vee_{p=0}^{s} g_1(t_2, p)\right)$$

$$+ \aleph_{f1}(\epsilon) \int_0^{t_2} n_1(t_2, s)\varphi_1(|\, y(s)|) d_s\left(\vee_{p=0}^{s} g_1(t_2, p)\right)$$

$$+[m_1\,(t_1)|y(t_1)| + |f_1(t_1, 0)|] \int_0^{t_2} n_1(t_2, s)\varphi_1(|\, y(s)|) d_s\left(\vee_{p=0}^{s}[g_1(t_2, p) - g_1(t_1, p)]\right)$$

$$+ [m_1\,(t_1)|y(t_1)| + |f_1(t_1, 0)|] \int_0^{t_2} \aleph_{u1}(\epsilon) d_s\left(\vee_{p=0}^{s} g_1(t_1, p)\right)$$

$$+[m_1\,(t_1)|y(t_1)| + |f_1(t_1, 0)|] \int_{t_1}^{t_2} n_1(t_1, s)\varphi_1(|\, y(s)|) d_s\left(\vee_{p=0}^{s} g_1(t_1, p)\right)$$

Where

$$\aleph(h_i, \epsilon) = sup\,\{|h_1(t_2) - h_1(t_1)|:\ t_1, t_2 \in\ I, |t_2\ -\ t_1| <\ \epsilon, i = 1, 2\,\},$$

$$\aleph_{fi}(\epsilon) = sup\,\{|f_i(t_2, v) - f_i(t_1, v)| :\ \ t_1, t_2 \in\ I, |t_2\ -\ t_1| <\ \epsilon, v \in R, i = 1, 2\},$$

$$\aleph_{ui}(\epsilon) = \left\{\left|u_1\left(t_2, s, v(s)\right) - u_2(t_1, s, v(s))\right| :\ t_1, t_2 \in\ I, |t_2\ -\ t_1| <\ \epsilon, v \in R, i = 1, 2\,\right\}.$$

Then, form estimate we get

$$|T_1 y(t_2) - T_1 y(t_1)| \leq\ \aleph(h_1, \epsilon) + [m_1(t_2)\aleph(y, \epsilon)\aleph_{fi}(\epsilon)]n_1\varphi_1(\|\,|\, y|\|) + \int_0^{t_2} d_s\left(\vee_{p=0}^{s} g_1(t_2, p)\right)$$

$$+ [m_1 \|y\| + F_1]\Big[n_1\varphi_1(\||\, y|\|)\int_0^{t_2} d_s\big(\vee_{p=0}^{s}[g_1(t_2,p) - g_1(t_1,p)]\big)$$

$$+ \aleph_{u1}(\epsilon)\int_0^{t_2} d_s\big(\vee_{p=0}^{s} g_1(t_2,p)\big) + n_1\varphi_1(\||\, y|\|)\int_{t_1}^{t_2} d_s\big(\vee_{p=0}^{s} g_1(t_1,p)\big)\Big]$$

$$\leq \aleph(h_1,\epsilon) + [m_1(t_2)\aleph(y,\epsilon)\aleph_{fi}(\epsilon)]n_1\varphi_1(\||\, y|\|) + \int_0^{t_2} d_s(\vee_{s=0}^{t} g_1(t_2,s))$$

$$+ [m_1 \|y\| + F_1]\Big[n_1\varphi_1(\||\, y|\|)\int_0^{t_2} d_s(\vee_{s=0}^{t}[g_1(t_2,s) - g_1(t_1,s)])$$

$$+ \aleph_{u1}(\epsilon)\int_0^{t_2} d_s(\vee_{s=0}^{s} g_1(t_2,s)) + n_1\varphi_1(\||\, y|\|)\int_{t_1}^{t_2} d_s(\vee_{s=0}^{t} g_1(t_1,s))\Big]$$

$$\leq \aleph(h_1,\epsilon) + K_1[m_1\aleph(y,\epsilon) + \aleph_{f1}(\epsilon)]n_1\varphi_1(r)$$

$$+ [m_1\, r + F_1]\big[n_1\varphi_1(r)W_1(\epsilon) + \aleph_{f1}(\epsilon)[g_1(t_1,t_2) - g_1(t_1,0)]$$

$$+ n_1\varphi_1(r)[g_1(t_1,t_2) - g_1(t_1,t_1)]\big].$$

Hence, from the continuity of the functions g_1 supposition (vi), we conclude that T_1 maps $C(I)$ into $C(I)$. As done above we can gain

$$|T_2y(t_2) - T_2y(t_1)| \leq \aleph(h_2,\epsilon) + K_2[m_2\aleph(y,\epsilon) + \aleph_{f2}(\epsilon)]n_2\varphi_2(r)$$

$$+ [m_2\, r + F_2]\big[n_2\varphi_2(r)W_2(\epsilon) + \aleph_{f2}(\epsilon)[g_2(t_1,t_2) - g_2(t_1,0)]$$

$$+ n_2\varphi_2(r)[g_2(t_1,t_2) - g_2(t_1,t_1)]\big]$$

Also, by our supposition (iv), we see that T_2 maps $C(I)$ into $C(I)$.

Now, from the definition of the operator T we get

$$Tv(t_2) - Tv(t_1) = T(x,y)(t_2) - T(x,y)(t_1)$$

$$= \left(T_1 y(t_2), T_2 x(t_2)\right) - (T_1 y(t_1), T_2 x(t_1))$$

$$= \left(T_1 y(t_2) - T_1 y(t_1), T_2 x(t_2) - T_2 x(t_1)\right).$$

Thus

$$\|Tv(t_2) - Tv(t_1)\| = \left\|\left(T_1 y(t_2) - T_1 y(t_1), T_2 x(t_2) - T_2 x(t_1)\right)\right\|$$

$$= max\,\{\|T_1 y(t_2) - T_1 y(t_1)\|,\ \|T_2 y(t_2) - T_2 y(t_1)\|\}$$

$$\leq max\,\{\aleph(h_1,\epsilon) + K_1[m_1\aleph(y,\epsilon) + \aleph_{f1}(\epsilon)]n_1\varphi_1(r)$$

$$+ [m_1 r + F_1][n_1\varphi_1(r)W_1(\epsilon) + \aleph_{u1}(\epsilon)[g_1(t_1,t_2) - g_1(t_1,0)]$$

$$+ n_1\varphi_1(r)[g_1(t_1,t_2) - g_1(t_1,t_1)]],$$

$$\aleph(h_2,\epsilon) + K_2[m_2\aleph(y,\epsilon) + \aleph_{f2}(\epsilon)]n_2\varphi_2(r)$$

$$+ [m_2 r + F_2][n_2\varphi_2(r)W_2(\epsilon) + \aleph_{u2}(\epsilon)[g_2(t_1,t_2) - g_2(t_1,0)]$$

$$+ n_2\varphi_2(r)[g_2(t_1,t_2) - g_2(t_1,t_1)]]\}.$$

This means that the class of $\{Tv(t)\}$ is equi-continuous on I, then by Arzela-Ascoil theorem $\{Tv(t)\}$ is relatively compact.

Now, we will show that the operator $T : V \to V$ is continuous.

originally, we prove that T_1 is continuous. Let $\epsilon * > 0$, the continuity of $u_i, i = 1, 2,$ yields, $\exists\, \delta = \delta(\epsilon *)$ such that $\left|u_i(t,s,u(s)) - u_i(t,s,v(s))\right| < \epsilon *$, whenever $\| u - v \| \leq \delta$, therefore if $\| y - u \| \leq \delta$, we arrive at:

$$|(T_1 y)(t) - (T_1 u)(t)|$$

$$\leq \Big| f_1(t, y(\omega_1(t))) \int_0^t u_1(t,s,y(\omega_1(s)) d_s g_1(t,s)$$

$$- f_1(t, u(\omega_1(t))) \int_0^t u_1(t,s,u(\omega_1(s)) d_s g_1(t,s) \Big|$$

$$\leq \Big| f_1(t, y(\omega_1(t))) \int_0^t u_1(t,s,u(\omega_1(s)) d_s g_1(t,s)$$

$$- f_1(t, u(\omega_1(t)) \int_0^t u_1(t, s, y(\omega_1(s)) d_s g_1(t, s) \Big|$$

$$+ f_1(t, u(\omega_1(t)) \int_0^t u_1(t, s, y(\omega_1(s)) d_s g_1(t, s)$$

$$- f_1(t, u(\omega_1(t)) \int_0^t u_1(t, s, u(\omega_1(s)) d_s g_1(t, s) \Big|$$

$$\leq \left| f_1(t, y(\omega_1(t)) - f_1(t, u(\omega_1(t)) \right| \int_0^t \left| u_1(t, s, y(\omega_1(s)) \right| d_s g_1(t, s)$$

$$+ |f_1(t, u(\omega_1(t))| \int_0^t \left| u_1(t, s, y(\omega_1(s)) - u_1(t, s, u(\omega_1(s)) \right| d_s g_1(t, s)$$

$$\leq m_1(t)|y(t) - u(t)| \int_0^t n_1(t, s) \varphi_1(|y(s)|) \; d_s g_1(t, s)$$

$$+ [m_1(t)|u(t_1)| + |f_1(t_1, 0)|] \int_0^t \left| u_1(t, s, y(\omega_1(s)) - u_1(t, s, u(\omega_1(s)) \right| d_s g_1(t, s)$$

$$\leq (\delta m_1 n_1 \varphi_1(\|y\|) + [m_1\|u\| + F_1]\epsilon^*) \int_0^t d_s \bigvee_{p=0}^{s} g_1(t, p)$$

$$\leq (\delta m_1 n_1 \varphi_1(\|y\|) + [m_1\|u\| + F_1]\epsilon^*) \bigvee_{s=0}^{t} g_1(t, s),$$

$$\leq (\delta m_1 n_1 \varphi_1(\|y\|) + [m_1\|u\| + F_1]\epsilon^*) K_1,$$

where

$$\epsilon = (\delta m_1 n_1 \varphi_1(\|y\|) + [m_1\|u\| + F_1]\epsilon^*) K_1.$$

Therefore

$$|(T_1 y)(t) - (T_1 u)(t)| \leq \epsilon.$$

This means that the operator T_1 is continuous.

By a analogous way as done above we can prove that for any $x, u \in C[0, T]$ and $\| x - v \| < \delta$, we have

$$|(T_2 x)(t) - (T_2 v)(t)| \leq \epsilon.$$

Hence T_2 is continuous operator. The operators T_i $(i = 1, 2)$ are continuous operator this indicate that T is continuous operator. Since all conditions of

Schauder fixed point theorem are satisfied, also T has at least one fixed point $v = (x, y) \in V$, which completes the evidence.

Corollary 1. Let hypotheticals of Theorem 1 be satisfied. Then quadratic Volterra-Stieltjes functional integral equation

$$x(t) = h(t) + f(t, y(\omega(t))) \int_0^t u(t, s, y(\omega(s)) d_s g(t, s), \ t \in I \qquad \textbf{(3)}$$

has at least one solution $x \in C(I)$.

Proof. Let the hypotheticals of Theorem 1 be satisfied. With $x = y$, $f_1 = f_2 = f$, $u_1 = u_2 = u$, and $h_1 = h_2 = h$. Also the coupled system (1) will be the Volterra-Stieltjes quadratic integral equation (3)

3. Existence of unique solution

Here, we study the uniqueness of the solution $(x, y) \in X$ of the coupled system of

quadratic Volterra-Stieltjes integral equations (1). Assume that functions $\varphi_i : R+ \to R+$ have the

form $\varphi_i(x) = 1 + |x|$, and the functions $n_i(t, s) \in C(I)$ denoted by

$b_i = \| n_i \| = max\{n_i(t, s)\ t, s \in I, i = 1,2\}$. Then

$$|u_i(t, s, x)| \leq n_i(t, s)(1 + |x|).$$

Notice that this assumption is a special case of assumption (iii).

Consider now the assumptions (ii)*, (iii)* having the form

(ii)* $f_i: I \times R \to R$ are continuous functions and there exist constants numbers m_i such that

$$|f_i(t,x) - f_i(t,y)| \leq m_i|x-y|, \qquad i = 1,2.$$

From this assumption, we can deduce that

$$|u_i(t,s,x)| - |u_i(t,s,0)| \leq |u_i(t,s,x) - u_i(t,s,0)| \leq b_i|x|,$$

which implies that

$$|u_i(t,s,x)| \leq |u_i(t,s,0)| + b_i|x| \leq n_i(t,s) + b_i|x|,$$

where $n_i(t,s) = \sup_{t\in I}|u_i(t,s,0)|$.

Theorem 2. Let hypotheticals of Theorem 1 be satisfied with replace hypotheticals (ii), (iii) by (ii)*, (iii)*, if the following conditions hold

$$m(n+br) + (mr+F)K \leq 1.$$

Then the coupled system (1) has an unique solution $(x,y) \in X$.

Proof. Let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ be two solutions of the coupled system (1), we have

$$\|(x_1,y_1) - (x_2,y_2)\|_X = \|(x_1 - x_2, y_1 - y_2\|_X$$

$$= \max_{t\in I}\{\|x_1 - x_2\|, \|y_1 - y_2\|\}.$$

Now

$$|x_1(t) - x_2(t)|$$

$$\leq \left|f_1(t, y_1(\omega_1(t)) \int_0^t u_1(t,s,y_1(\omega_1(s))d_s g_1(t,s)\right.$$

$$-f_1(t,y_2(\omega_1(t))\int_0^t u_1(t,s,y_2(\omega_1(s))d_s g_1(t,s)\Big|$$

$$\leq \Big|f_1(t,y_1(\omega_1(t))\int_0^t u_1(t,s,y_1(\omega_1(s))d_s g_1(t,s)$$
$$-f_1(t,y_2(\omega_1(t))\int_0^t u_1(t,s,y_1(\omega_1(s))d_s g_1(t,s)\Big|$$
$$+\Big|f_1(t,y_2(\omega_1(t))\int_0^t u_1(t,s,y_1(\omega_1(s))d_s g_1(t,s)$$
$$-f_1(t,y_2(\omega_1(t))\int_0^t u_1(t,s,y_2(\omega_1(s))d_s g_1(t,s)\Big|$$
$$\leq \big|f_1(t,y_1(\omega_1(t)\big)-f_1(t,y_2(\omega_1(t))\big|\int_0^t\big|u_1(t,s,y_1(\omega_1(s))\big|d_s g_1(t,s)$$
$$+|f_1(t,y_2(\omega_1(t))|\int_0^t\big|u_1(t,s,y_1(\omega_1(s))-u_1(t,s,y_2(\omega_1(s))\big|d_s g_1(t,s)$$
$$\leq m_1|y_1(t)-y_2(t)|\int_0^t(n_1(t,s)+b_1|y|)d_s g_1(t,s)$$
$$+[m_1|y_2(t)|+|f_1(t,0)|]b_i\int_0^t|y_1(t)-y_2(t)|\,d_s g_1(t,s)$$
$$\leq [\|y_1-y_2\|m_1(n_1-b_1\|y_1\|)+[m_1\|y_2\|+F_1]\|y_1-y_2\|\int_0^t d_s(\vee_{p=0}^{s} g_1(t,p))$$
$$\leq m_1(n_1+b_1\|y_1\|)+[m_1\|y_2\|+F_1\|y_1-y_2\|](\vee_{s=0}^{t} g_1(t,s))$$
$$\leq m_1(n_1+b_1r_1)+[m_1r_1+F_1]K_1\|y_1-y_2\|.$$

Therefore

$$\|x_1-x_2\|\leq m(n+br)+[mr+F]K\|y_1-y_2\|,$$

where

$b=max\{b_1,b_2\},\ m=max\{m_1,m_2\},\ n=max\{n_1,n_2\},\ F=max\{F_1,F_2\}$ and

$$K=max\,\{K_1,K_2\}.$$

Similarly

$$\|y_1-y_2\|\leq m(n+br)+[mr+F]K\|x_1-x_2\|.$$

Then

$$\|(x_1,y_1)-(x_2,y_2)\|_X=\|(x_1-x_2,y_1-y_2\|_X$$
$$=\max_{t\in I}\{\|x_1-x_2\|_C,\|y_1-y_2\|_C\}$$
$$=\max_{t\in I}\{m(n+br)+[mr+F]K\|y_1-y_2\|m(n+br)+[mr+F]K\|x_1-x_2\|\}$$
$$=[m(n+br)+[mr+F]K]\max_{t\in I}\{\|x_1-x_2\|_C,\|y_1-y_2\|_C\}$$
$$=m(n+br)+[mr+F]K\ \|(x_1,y_1)-(x_2,y_2)\|_X.$$

Which implies that

$$[1-m(n+br)+[mr+F]K]\|(x_1,y_1)-(x_2,y_2)\|_X\leq 0.$$

This means that

$$(x_1,y_1)=(x_2,y_2)\ \Rightarrow\ x_1=x_2,\qquad y_1=y_2.$$

This proves the uniqueness of the solution of the coupled system (1).

4. SPECIAL CASES

A related system of fractional-order quadratic Volterra integral equations of the following type will be studied in this section:

$$x(t) = h_1(t) + f_1(t, y(\omega_1(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha_1)} u_1(t, s, y(\omega_1(t)))\, ds, \quad t \in I$$

$$(4)$$

$$y(t) = h_1(t) + f_2(t, x(\omega_2(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha_1)} u_2(t, s, x(\omega_2(t)))\, ds, \qquad t \in I$$

where $t \in I = [0, T]$ and $\alpha_i \in (0, 1)$, and $\Gamma(\alpha_i), i = 1, 2$, refers to gamma functions. Let us mention that (4) represents the so-called a coupled systems of Volterra quadratic integral equations of fractional order. Recently, such a type this type has been widely investigated in some papers [1, 9, 10, 12, 13, 14]

Here, we show that a coupled systems of fractional orders (4) can be treated as a special case of a coupled systems of quadratic Volterra-Stieltjes integral equations (2) studied in Section 2.

In fact, we can consider functions $g_i(t, s) = g_i : \triangle \to R, i = 1, 2$, defined by the formula

$$g_i(t, s) = \frac{t^{\alpha_i} - (t-s)^{\alpha_i}}{\Gamma(\alpha_i + 1)}.$$

We can see that functions $g_i, i = 1, 2$, satisfy hypotheticals (vi)-(vii) in Theorem 1, see [6, 8].

Now, we state the following existence results for couple system of quadratic Volterra integral equations of fractional order (4).

Theorem 3. Let hypotheticals (i)-(viii) of Theorem 1 be satisfied. Then a coupled systems of fractional orders (4) has at least one solution $(x, y) \in X$.

Corollary 2. Let hypotheticals of Theorem 3 be satisfied (with $= y, u_1 = u_2 = u, f_1 = f_2 = f, h_1 = h_2 = h$ and $\alpha_1 = \alpha_2 = \alpha$). Then the fractional-order quadratic integral equation

$$x(t) = h(t) + f(t, x(\omega(t))) \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(t, s, x(\omega(t)))\, ds, \; t \in I$$

has at least one solution in x ∈ C(I) .

Corollary 3. Let hypotheticals of Theorem 3 be satisfied, with $f_1(t, y(t)) = f_2(t, x(t)) = 1$. Then a coupled system of the fractional-order quadratic integral system

$$x(t) = h_1(t) + \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_1(t, s, y(\omega_1(t)))\, ds, \qquad t \in I$$

(5)

$$y(t) = h_1(t) + \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} u_2(t, s, x(\omega_2(t)))\, ds, \qquad t \in I$$

has at least one solution in $(x, y) \in X$.

Now, letting $\alpha_1, \alpha_2 \to 1$, we obtain

Corollary 4. Let hypotheticals of Theorem 3 be satisfied. Then the coupled system of the initial value problems

$$\frac{x(t)}{dt} = u_1\left(t, s, y\big(\omega_1(t)\big)\right), \qquad t \in I, \quad x(0) = x_0,$$

(6)

$$\frac{y(t)}{dt} = u_1\left(t, s, x\big(\omega_2(t)\big)\right), \qquad t \in I, \quad y(0) = y_0,$$

Proof. Let hypotheticals of Theorem 3 be satisfied (with $f_1\big(t, y(t)\big) = f_2\big(t, x(t)\big) = 1, h_1(t) = x_0,\; h_2(t) = y_0$ and letting $\alpha, \beta \to 1$. Then a coupled system of the fractional-order quadratic integral equations

$$x(t) = x_0 + \int_0^t u_1(t,s,y(\omega_1(t)))\,ds, \qquad t \in I,$$

(7)

$$y(t) = y_0 + \int_0^t u_1(t,s,x(\omega_2(t)))\,ds, \qquad t \in I,$$

has at least one solution in X which is equivalent to the coupled system of the initial value problems (6).

Corollary 5. Let hypotheticals of Theorem 3 be satisfied. Then the coupled system of fractional order differential equations

$$D^{\alpha_1}x(t) = u_1\left(t,s,y\left(\omega_1(t)\right)\right), \qquad t \in I$$

(8)

$$D^{\alpha_2}y(t) = u_2\left(t,s,x\left(\omega_2(t)\right)\right), \qquad t \in I,$$

with the initial conditions

$$I^{1-\alpha_1}x(t)\,\big|_{t=0} = I^{1-\alpha_2}y(t)\,\big|_{t=0} = 0,\ \ \alpha_1,\alpha_2 \in (0,1], \qquad (9)$$

has at least one solution in $(x,y) \in X$.

Proof. let us proof the coupled system of the initial value problems (8) and (9) is equivalent to the coupled system of quadratic integral system

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, u_1(t,s,y(\omega_1(t)))\,ds, \qquad t \in I$$

(10)

$$y(t) = \int_o^t \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} u_2(t,s,x(\omega_2(t)))\, ds, \qquad t \in I$$

By operating $I^{1-\alpha_1}$ and $I^{1-\alpha_2}$respectively on each equation of coupled system (10), and applying properties of fractional operator [23], we obtain

$$I^{1-\alpha_1}x(t) = I^1 u_1(t,s,y(\omega_1(t))), \qquad I^{1-\alpha_1}x(t)\,\big|_{t=0} = 0$$

$$I^{1-\alpha_2}y(t) = I^1 u_2(t,s,y(\omega_2(t))),\ I^{1-\alpha_2}y(t)\,\big|_{t=0} = 0.$$

Also,

$$\frac{d}{dt} I^{1-\alpha_1}x(t) = u_1\Big(t,s,y\big(\omega_1(t)\big)\Big),\ t \in I,\ \alpha_1 \in (0,1)$$

$$\frac{d}{dt} I^{1-\alpha_2}y(t) = u_2\Big(t,s,y\big(\omega_2(t)\big)\Big),\ t \in I,\ \alpha_2 \in (0,1).$$

Conversely, by integrating the coupled system of initial value problems (8) and (9), we have

$$I^{1-\alpha_1}x(t) - I^{1-\alpha_1}x(t)\,\big|_{t=0} = I^1 u_1(t,s,y(\omega_1(t)))$$

$$I^{1-\alpha_2}y(t) - I^{1-\alpha_2}y(t)\,\big|_{t=0} = I^1 u_2\Big(t,s,y\big(\omega_2(t)\big)\Big).$$

Operating by I^{α_1} and I^{α_2} respectively on each equation and differentiating, we have (10). Thus, the equivalence hold.

Let hypotheticals of Theorem 3 be satisfied (with $f_1\Big(t,y\big(\omega_1(t)\big)\Big) = f_2\Big(t,x\big(\omega_2(t)\big)\Big) = 1, h_1(t) = h_2(t) = 0$. Then there exists at least one solution in X for the coupled system (8 and 9).

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